

Hypergeometric solutions of the closed eigenvalue problem on Heisenberg Isoperimetric Profiles

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Abstract

After introducing the sub-Riemannian geometry of the Heisenberg group \mathbb{H}^n , $n \geq 1$, we recall some basics about hypersurfaces endowed with the H -perimeter measure σ_H^{2n} and horizontal Green's formulas. Then, we describe a class of compact closed hypersurfaces of constant horizontal mean curvature called "Isoperimetric Profiles" (they are not CC-balls!); see Section 2.1. Our main purpose is to study a closed eigenvalue problem on Isoperimetric Profiles, i.e. $\mathcal{L}_{HS}\varphi + \lambda\varphi = 0$, where \mathcal{L}_{HS} is a 2nd order horizontal tangential operator analogous to the Laplace-Beltrami operator; see Section 1.5. This is done starting from the radial symmetry of Isoperimetric Profiles with respect to a barycentric axis parallel to the center T of the Lie algebra \mathfrak{h}_n . An interesting feature of radial eigenfunctions is in that they are hypergeometric functions; see Theorem 2.10. Finally, in Section 2.3 we shall begin the study of the general case.

KEY WORDS AND PHRASES: Heisenberg Groups; Sub-Riemannian Geometry; Hypergeometric Solutions; Closed Eigenvalue Problem; Isoperimetric Profiles

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1 Introduction

In the last decades, sub-Riemannian geometry has become a subject of great interest due to its connections with many different areas of Mathematics and Physics, such as PDE's, Calculus of Variations, Control Theory, Mechanics and, more recently, Visual Geometry (see [38] and references therein) and Theoretical Computer Science (see [7]). For references and comments we refer the reader to the seminal paper by Gromov, [17], Montgomery's book, [31], and the papers by Pansu, [35], Strichartz, [41], and Vershik and Gershkovich, [43].

In the context of sub-Riemannian geometry, Carnot groups provide a very large class of examples. They are connected, simply connected, nilpotent and stratified¹ Lie groups which play in sub-Riemannian geometry a role analogous to that of Euclidean spaces in Riemannian geometry. A fundamental feature of Carnot groups is that they are homogeneous groups in the sense of Stein's definition (see [40]), and

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¹A vector space E is *stratified* if there exist vector subspaces E_1, \dots, E_k of E such that $E = E_1 \oplus \dots \oplus E_k$. The subspace E_i is called the *i-th layer* of the stratification while the integer k is the *step* of E .

this means that they are equipped with an intrinsic family of anisotropic dilations. The homogeneous dimension of any Carnot group is the weighted sum of the dimensions of each layer of the stratification and this integer equals the intrinsic metric dimension of the group, considered as a metric space with respect to the so-called *Carnot-Carathéodory* distance. For an introduction to the geometry of Lie groups we refer the reader to Helgason, [18], and Milnor, [27], while specifically for sub-Riemannian geometry to Gromov, [17], Pansu, [35], and Montgomery, [31]. Carnot groups nowadays represent an intensive research-field in Analysis and Geometric Measure Theory; see, for instance, [2], [4], [8], [10, 11], [15, 16], [21], [22], [26], [29], [36], [37], but these references are far from being complete.

Among Carnot groups, the most common ones are the so-called *Heisenberg groups* \mathbb{H}^n , $n \geq 1$; see Section 1.1. Roughly speaking, as a manifold, \mathbb{H}^n can be regarded as $\mathbb{C}^n \times \mathbb{R}$ endowed with a suitable polynomial group law \star . Its Lie algebra \mathfrak{h}_n identifies with $T_0\mathbb{H}^n$ (i.e. the tangent space at $0 \in \mathbb{H}^n$). Let us introduce a left invariant frame $\mathcal{F} = \{X_1, Y_1, \dots, X_n, Y_n, T\}$ for $T\mathbb{H}^n$, where $X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}$, $Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}$ and $T = \frac{\partial}{\partial t}$. Denoting by $[\cdot, \cdot]$ the Lie bracket of vector fields, one has

$$[X_i, Y_i] = T \quad \text{for every } i = 1, \dots, n,$$

and all other commutators vanish. In other words, \mathfrak{h}_n turns out to be nilpotent and stratified with center T , i.e. $\mathfrak{h}_n = H \oplus H_2$, where $H = \text{span}_{\mathbb{R}}\{X_1, Y_1, \dots, X_n, Y_n\}$ is the horizontal bundle and $H_2 = \text{span}_{\mathbb{R}}\{T\}$ is the vertical bundle. Later on, \mathbb{H}^n will be endowed with the left-invariant Riemannian metric $h := \langle \cdot, \cdot \rangle$ making \mathcal{F} an orthonormal frame. This metric induces a corresponding metric h_H on H which provides a way to measure the length of horizontal curves. In fact, the natural distance in sub-Riemannian geometry is the *Carnot-Carathéodory distance* d_{CC} , defined by minimizing the (Riemannian) length of all piecewise smooth horizontal curves joining two different points².

The stratification of \mathfrak{h}_n is connected with the existence of a 1-parameter group of automorphisms, called *Heisenberg dilations*, defined by $\delta_s(z, t) := (sz, s^2t)$, where $(z, t) \in \mathbb{R}^{2n+1}$ denote exponential coordinates of a point $p \in \mathbb{H}^n$. The integer $Q = 2n + 2$ turns out to be the (homogeneous) dimension of \mathbb{H}^n as a metric space with respect to the CC-distance d_{CC} .

Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth hypersurface. By definition, the *characteristic set* $C_S \subsetneq S$ is the set of all points $x \in S$ such that $\dim H_x = \dim(H_x \cap T_x S)$. In other words, $x \in S$ is non-characteristic if, and only if, the horizontal bundle H is transversal to TS at the point x . If S is non-characteristic, the *unit H -normal* along S is defined by $\nu_H := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}$, where ν is the Riemannian unit normal along S and $\mathcal{P}_H : T\mathbb{H}^n \rightarrow H$ is the orthogonal projection onto H . Moreover, there exists a homogeneous measure σ_H^{2n} on S , called *H -perimeter measure*, which can be defined in terms of the Riemannian measure $\sigma_{\mathbb{R}}^{2n}$ on S . More precisely, one has

$$\sigma_H^{2n} \llcorner S = |\mathcal{P}_H \nu| \sigma_{\mathbb{R}}^{2n} \llcorner S.$$

We have to remark that the measure σ_H^{2n} does not see characteristic points. Furthermore, denoting by $\mathcal{S}_{\mathbb{C}\mathbb{C}}^{Q-1}$ the $(Q-1)$ -dimensional spherical Hausdorff measure associated with d_{CC} , it turns out that $\sigma_H^{2n}(S \cap B) = k(\nu_H) \mathcal{S}_{\mathbb{C}\mathbb{C}}^{Q-1}(S \cap B)$ for all Borel set B , where $k(\nu_H)$ is a density-function called *metric factor*; see [26]. On $S \setminus C_S$, there are two important subbundles to be defined: the *horizontal tangent bundle* $HS \subset TS$ and the *horizontal normal bundle* $\nu_H S$. They split the horizontal bundle H into an orthogonal direct sum, i.e. $H = \nu_H S \oplus HS$. On the other hand, at $C_S \subsetneq S$, only the subbundle HS turns out to be defined, and in this case $HS = H$.

Following the somehow general geometric approach begun in [29], in this paper we address the issue of studying a closed eigenvalue problem for a 2nd order horizontal tangential operator \mathcal{L}_{HS} , which is the sub-Riemannian analogous to the Laplace-Beltrami operator. More precisely, let grad_{HS} denote the gradient operator on the horizontal tangent bundle HS . A simple way to define the operator \mathcal{L}_{HS} reads as follows:

$$\mathcal{L}_{HS} \varphi \sigma_H^{2n} \llcorner S := d(\text{grad}_{HS} \varphi \lrcorner \sigma_H^{2n}) \llcorner S$$

for all \mathbf{C}^2 -smooth function $\varphi : S \rightarrow \mathbb{R}$, where $\text{grad}_{HS} \varphi = \text{grad}_H \varphi - \langle \text{grad}_H \varphi, \nu_H \rangle \nu_H$. In this definition, $d : \Lambda^k(T^*\mathbb{H}^n) \rightarrow \Lambda^{k+1}(T^*\mathbb{H}^n)$ is the exterior derivative and $\lrcorner : \Lambda^k(T^*\mathbb{H}^n) \rightarrow \Lambda^{k-1}(T^*\mathbb{H}^n)$ is the “contraction operator” on differential forms; see [18], [13]. The importance of the operator \mathcal{L}_{HS} comes from some horizontal Green-type formulas discussed in Section 1.3; see also Section 1.4.

In a certain sense, what we are trying to do is to prove a sub-Riemannian generalization of the closed eigenvalue problem for the Laplacian on Spheres. Actually, we are interested to find explicit solutions to a “closed eigenvalue problem”³ for the operator \mathcal{L}_{HS} , in the special case of *Isoperimetric Profiles*. They are compact hypersurfaces which can be efficiently described in terms of a key-property of CC-geodesics.

²The last definition is motivated by Chow’s Theorem, which implies that two different points can be joined by (infinitely many) horizontal curves.

³This means solving the equation $\mathcal{L}_{HS} \varphi + \lambda \varphi = 0$ on a smooth compact (closed) hypersurface $S \subset \mathbb{H}^n$.

Let us sketch a picture in the 1st Heisenberg group \mathbb{H}^1 . Remind that any CC-geodesic γ in \mathbb{H}^1 is either a Euclidean (horizontal) line or a suitable infinite circular helix of constant slope and whose axis is parallel to the center T of the Lie algebra \mathfrak{h}_1 . In the second case, choose a point $\mathcal{S} \in \gamma$ and take the (positively oriented) T -line \mathcal{V} over this point. There exists a first consecutive point to \mathcal{S} , denoted by \mathcal{N} , which⁴ belongs to $\gamma \cap \mathcal{V}$. These points, henceforth called South and North poles, determine a minimizing connected subset of γ . The slope of this helix is uniquely determined by the CC-distance of the poles. Now rotating (around the vertical axis from \mathcal{N} to \mathcal{S}) the curve joining the poles yields a closed convex surface very similar to an ellipsoid, henceforth called *Isoperimetric Profile*. A similar description can be done in the general case. Hereafter, we shall denote by $\mathbb{S}_{\mathbb{H}^n}$ the unit Isoperimetric Profile of \mathbb{H}^n ; see Section 2.1.

We also stress that Isoperimetric Profiles are constant horizontal mean curvature playing in Heisenberg groups an equivalent role of spheres in Euclidean spaces. In this regard, there is a longstanding conjecture claiming that Isoperimetric Profiles minimize the H -perimeter measure among “finite H -perimeter sets”⁵ having fixed volume; see [4], [10], [11], [12], [21], [22, 23], [24], [25], [32], [33, 34], [37]. For this reason, it seems interesting to study some features of these sets from a sub-Riemannian point of view.

The plan of the paper is the following. After introducing in Section 1.1 the sub-Riemannian geometry of Heisenberg groups \mathbb{H}^n , $n \geq 1$, we overview some basic facts concerning the theory of hypersurfaces endowed with the H -perimeter measure σ_H^{2n} . We also prove some useful geometric identities for the sequel; see Section 1.2. In Section 1.3 and Section 1.4 we discuss some horizontal divergence formulas. In Section 2.1 we briefly describe Heisenberg Isoperimetric Profiles. In Section 1.5 we examine some general features of the closed eigenvalue problem for the operator \mathcal{L}_{HS} on compact closed hypersurfaces, i.e.

$$\begin{cases} \mathcal{L}_{HS} \varphi = -\lambda \varphi & \text{on } S \setminus C_S \\ \int_S \varphi \sigma_H^{2n} = 0; \end{cases}.$$

see Problem 1.29. As already said, our main interest concerns the case of Isoperimetric Profiles and so we shall choose $S = \mathbb{S}_{\mathbb{H}^n}$.

Remark 1.1. Let C_P denote the best constant in the following Poincaré-type inequality

$$\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi^2 \sigma_H^{2n} \leq C_P \int_{\mathbb{S}_{\mathbb{H}^n}} |\text{grad}_{HS} \varphi|^2 \sigma_H^{2n}$$

for all φ belonging to the horizontal Sobolev space $\mathcal{H}(\mathbb{S}_{\mathbb{H}^n}, \sigma_H^{2n})$ such that $\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi \sigma_H^{2n} = 0$. Then, it turns out that

$$C_P = \frac{1}{\mu},$$

where μ is the 1st eigenvalue of Problem 1.29; see Section 1.5.

Our starting point is the radial symmetry of Isoperimetric Profiles with respect to a barycentric axis parallel to the center T of \mathfrak{h}_n ; see Section 2.2. In this way, we reduce ourselves to a *radial eigenvalue problem*; see Problem 2.4. Therefore, we have to study an O.D.E. of hypergeometric type, which can be explicitly integrated by classical methods for hypergeometric equations. The spectrum of \mathcal{L}_{HS} on radial functions is found by using some natural mixed conditions for this problem. Eigenfunctions turn out to be hypergeometric functions; see Theorem 2.10. Then, after studying the radial case, in Section 2.3 we overview some features of the general case. Among other things, we prove that the spectrum of \mathcal{L}_{HS} contains the radial spectrum; see Proposition 2.22. This is done by showing that the spherical mean of each eigenfunction of \mathcal{L}_{HS} associated with an eigenvalue λ (whenever different from 0) must be an eigenfunction of the radial eigenvalue problem associated with λ . This fact raises the following question (see e.g. Remark 2.24):

Is the 1st eigenvalue of Problem 1.29 equal to the first eigenvalue of Problem 2.4?

Finally, by applying a standard trick (see [14]) we will prove some related inequalities; see Corollary 2.26 and Corollary 2.24. These inequalities are stronger than what one might expect and this fact seems to suggest a positive answer to the previous question.

⁴This point can be interpreted as the cut point of \mathcal{S} along γ . In fact this is the end-point of all CC-geodesics starting from \mathcal{S} with same slope. Note however that, strictly speaking, the cut locus of any point in \mathbb{H}^n ($n \geq 1$) coincides with the vertical T -line over the point.

⁵In the variational sense; for a definition, see [15].

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This paper is the first of a series devoted to study, in the context of Heisenberg groups, constant horizontal mean curvature hypersurfaces and among them, Isoperimetric Profiles. In many respects, this project begun during my visit at Purdue University in November 2006. So I wish to express my gratitude to Prof. N. Garofalo for his kind invitation and for many interesting conversations during the last years. Furthermore, I would like to thank Prof. A. Parmeggiani for his sincere interest in this work and C. Senni for his help with MATHEMATICA.

1.1 Heisenberg group \mathbb{H}^n

The n -th Heisenberg group (\mathbb{H}^n, \star) , $n \geq 1$, is a connected, simply connected, nilpotent and stratified Lie group of step 2 on \mathbb{R}^{2n+1} , with respect to a polynomial group law \star ; see below. Its Lie algebra \mathfrak{h}_n is isomorphic to the $(2n+1)$ -dimensional real vector space \mathbb{R}^{2n+1} and it will be identified with the tangent space at the identity 0 of \mathbb{H}^n , i.e. $\mathfrak{h}_n \cong T_0\mathbb{H}^n$. We shall adopt *exponential coordinates of the 1st kind*. Hence every point $p \in \mathbb{H}^n$ identifies with an ordered $(2n+1)$ -tuple of the Lie algebra \mathfrak{h}_n , i.e. $p = \exp(x_1, y_1, \dots, x_n, y_n, t)$. For simplicity, $(z, t) \in \mathbb{R}^{2n+1}$ will denote exponential coordinates of a generic point $p \in \mathbb{H}^n$. In order to describe the algebraic structure of \mathfrak{h}_n , let us fix a global left-invariant frame $\mathcal{F} := \{X_1, Y_1, \dots, X_n, Y_n, T\}$ for $T\mathbb{H}^n$, where we have set

$$X_i(p) := \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}, \quad Y_i(p) := \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t} \quad (i = 1, \dots, n), \quad T(p) := \frac{\partial}{\partial t},$$

for every $p \in \mathbb{H}^n$. Denoting by $[\cdot, \cdot]$ the Lie bracket of vector fields, one has $[X_i, Y_i] = T$ for every $i = 1, \dots, n$, and all other commutators vanish. Hence, \mathfrak{h}_n turns out to be a nilpotent and stratified Lie algebra of step 2 with center T , i.e. $\mathfrak{h}_n = H \oplus H_2$, where $H := \text{span}_{\mathbb{R}}\{X_1, Y_1, \dots, X_n, Y_n\}$ and $H_2 = \text{span}_{\mathbb{R}}\{T\}$. The first layer H is called *horizontal*, whereas the second layer, spanned by T , is called *vertical*; they are both smooth subbundles of $T\mathbb{H}^n$.

The *Baker-Campbell-Hausdorff formula* uniquely determines the group law \star of \mathbb{H}^n , starting from the “structure” of its Lie algebra \mathfrak{h}_n ; see [9]. More precisely, we have

$$\exp X \star \exp Y = \exp(X \diamond Y) \quad \text{for every } X, Y \in \mathfrak{h}_n,$$

where $\diamond : \mathfrak{h}_n \times \mathfrak{h}_n \rightarrow \mathfrak{h}_n$ is the operation defined by $X \diamond Y := X + Y + \frac{1}{2}[X, Y]$. Therefore, for every $p = \exp(x_1, y_1, \dots, x_n, y_n, t)$, $p' = \exp(x'_1, y'_1, \dots, x'_n, y'_n, t')$ in \mathbb{H}^n , it follows that

$$p \star p' := \exp \left(x_1 + x'_1, y_1 + y'_1, \dots, x_n + x'_n, y_n + y'_n, t + t' + \frac{1}{2} \sum_{i=1}^n (x_i y'_i - x'_i y_i) \right). \quad (1)$$

Hence, the inverse of any $p \in \mathbb{H}^n$ is $p^{-1} := \exp(-x_1, -y_1, \dots, -x_n, -y_n, -t)$ and $0 = \exp(0_{\mathbb{R}^{2n+1}})$.

Definition 1.2. A sub-Riemannian metric h_H is a symmetric positive bilinear form on the horizontal bundle $H \subset T\mathbb{H}^n$. The CC-distance $d_{CC}(p, p')$ between $p, p' \in \mathbb{H}^n$ is defined by

$$d_{CC}(p, p') := \inf \int \sqrt{h_H(\dot{\gamma}, \dot{\gamma})} dt,$$

where the infimum is taken over all piecewise-smooth horizontal curves γ joining p to p' .

By Chow's Theorem, every couple of points can be connected by a horizontal curve (not unique, in general) and this implies that d_{CC} is a metric on \mathbb{H}^n . The topology generated by the CC-metric turns out to be equivalent to the standard topology on \mathbb{R}^{2n+1} ; see [17], [31].

From now on, we will equip $T\mathbb{H}^n$ with the left-invariant Riemannian metric $h := \langle \cdot, \cdot \rangle$ making \mathcal{F} an orthonormal frame and assume $h_H := h|_H$.

The *structural constants*⁶ of the Heisenberg Lie algebra \mathfrak{h}_n are completely described by the skew-

⁶Let $\mathfrak{g} \cong \mathbb{R}^n$ be a Lie algebra with respect to $[\cdot, \cdot]$, let $\langle \cdot, \cdot \rangle$ be a Euclidean metric on \mathfrak{g} and let $\mathcal{F} = \{X_1, \dots, X_n\}$ be an o.n. basis of \mathfrak{g} . The *structural constants* of \mathfrak{g} associated with \mathcal{F} are n^3 smooth functions defined by

$$C_{ij}^r := \langle [X_i, X_j], X_r \rangle \quad i, j, r = 1, \dots, n.$$

The structural constants embody all algebraic features of \mathfrak{g} ; see [18]. They satisfy the following properties:

- (i) $C_{ij}^r + C_{ji}^r = 0$ (Skew-symmetry)
- (ii) $\sum_{j=1}^n C_{ji}^i C_{rm}^j + C_{jm}^i C_{lr}^j + C_{jr}^i C_{ml}^j = 0$ (Jacobi Identity).

symmetric $(2n \times 2n)$ -matrix

$$C_H^{2n+1} := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 0 \end{bmatrix}.$$

We remark that C_H^{2n+1} is the matrix associated with the skew-symmetric bilinear map $\Gamma_H : H \times H \rightarrow \mathbb{R}$ defined by $\Gamma_H(X, Y) = \langle [X, Y], T \rangle$.

For every $p \in \mathbb{H}^n$, we denote by $L_p : \mathbb{H}^n \rightarrow \mathbb{H}^n$ the *left translation by p* , i.e. $L_p p' := p \star p'$. The map L_p is a group homomorphism and its differential $L_{p*} : T_0 \mathbb{H}^n \rightarrow T_p \mathbb{H}^n$ is given by

$$L_{p*} = \frac{\partial(p \star p')}{\partial p'} \Big|_{p'=0} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & 0 \\ -\frac{y_1}{2} & +\frac{x_1}{2} & \dots & -\frac{y_i}{2} & +\frac{x_i}{2} & \dots & -\frac{y_n}{2} & +\frac{x_n}{2} & 1 \end{bmatrix}.$$

Equivalently, one has $L_{p*} = \text{col}[X_1(p), Y_1(p), \dots, X_n(p), Y_n(p), T(p)]$. A key feature of \mathbb{H}^n is that there exists a 1-parameter group of automorphisms $\delta_s : \mathbb{H}^n \rightarrow \mathbb{H}^n$ ($s \geq 0$), hereafter called *Heisenberg dilations*, defined by $\delta_s p := \exp(sz, s^2 t)$ for every $s \geq 0$, where $p = \exp(z, t)$. As already said, the *homogeneous dimension* of \mathbb{H}^n with respect to the intrinsic dilations is the integer given by $Q := 2n + 2$ which equals the *Hausdorff dimension* of \mathbb{H}^n as a metric space with respect to the CC-distance d_{CC} ; see also [31].

We shall denote by ∇ the unique *left-invariant Levi-Civita connection* on \mathbb{H}^n associated with the metric $h = \langle \cdot, \cdot \rangle$. We recall that, for every $X, Y, Z \in \mathfrak{X} := \mathbf{C}^\infty(\mathbb{H}^n, T\mathbb{H}^n)$ one has

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle);$$

for more details, see [27]. For every $X, Y \in \mathfrak{X}_H := \mathbf{C}^\infty(\mathbb{H}^n, H)$, we shall set $\nabla_X^H Y := \mathcal{P}_H(\nabla_X Y)$, where \mathcal{P}_H denotes the orthogonal projection onto H . The operation ∇^H is a vector-bundle connection, called *H-connection*; see [29] and references therein. It is not difficult to show that ∇^H is *flat* and *compatible with the sub-Riemannian metric h_H* , i.e. $X \langle Y, Z \rangle = \langle \nabla_X^H Y, Z \rangle + \langle Y, \nabla_X^H Z \rangle$ for all $X, Y \in \mathfrak{X}_H$. Furthermore, ∇^H is *torsion-free*, i.e. $\nabla_X^H Y - \nabla_Y^H X - \mathcal{P}_H[X, Y] = 0$ for all $X, Y \in \mathfrak{X}_H$. All these properties follow from the very definition of ∇^H by using standard features of the Levi-Civita connection ∇ on \mathbb{H}^n .

Definition 1.3. For every $\psi \in \mathbf{C}^\infty(\mathbb{H}^n)$ the *H-gradient of ψ* is the unique horizontal vector field $\text{grad}_H \psi \in \mathfrak{X}_H$ such that $\langle \text{grad}_H \psi, X \rangle = d\psi(X) = X\psi$ for all $X \in H$. The *H-divergence $\text{div}_H X$* of any $X \in \mathfrak{X}_H$ is given, at each point $p \in \mathbb{H}^n$, by $\text{div}_H X(p) := \text{Trace}(Y \rightarrow \nabla_Y^H X)(p)$ ($Y \in H_p$). The *H-Laplacian Δ_H* is the 2nd order differential operator defined by

$$\Delta_H \psi := \text{div}_H(\text{grad}_H \psi) \quad \text{for every } \psi \in \mathbf{C}^\infty(\mathbb{H}^n).$$

We now recall some basic notions about left invariant forms on \mathbb{H}^n . Starting from the left invariant frame \mathcal{F} and by duality⁷, we define a global coframe $\mathcal{F}^* := \{X_1^*, Y_1^*, \dots, X_i^*, Y_i^*, \dots, X_n^*, Y_n^*, T^*\}$ of *left invariant 1-forms* for the cotangent bundle $T^*\mathbb{H}^n$, where

$$X_i^* = dx_i, \quad Y_i^* = dy_i \quad (i = 1, \dots, n), \quad T^* = dt + \frac{1}{2} \sum_{i=1}^n (y_i dx_i - x_i dy_i).$$

We shall set $\theta := T^*$ to denote the so-called *contact 1-form* of \mathbb{H}^n .

Finally the top-dimensional *left-invariant volume form* $\sigma_{\mathcal{R}}^{2n+1}$ of \mathbb{H}^n is defined by

$$\sigma_{\mathcal{R}}^{2n+1} := \left(\bigwedge_{i=1}^n dx_i \wedge dy_i \right) \wedge \theta.$$

The measure obtained by integration of $\sigma_{\mathcal{R}}^{2n+1}$ equals the *Haar measure* of \mathbb{H}^n .

⁷The duality is understood with respect to the fixed left-invariant metric h on \mathbb{H}^n . More generally, if (M, h) is a Riemannian manifold and $X \in TM$, then $X^*(Y) := h(X, Y)$ for every $Y \in TM$.

1.2 Hypersurfaces, measures and some useful formulas

Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^1 -smooth hypersurface and denote by ν its (Riemannian) unit normal. We have $\nu = \sum_{i=1}^n (\langle \nu, X_i \rangle X_i + \langle \nu, Y_i \rangle Y_i) + \nu_T T$, where

$$\begin{aligned} \langle \nu, X_i \rangle &= \frac{\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}}{\sqrt{\sum_{i=1}^n (\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2 + \langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2) + \langle T, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2}} \quad i = 1, \dots, n, \\ \langle \nu, Y_i \rangle &= \frac{\langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}}{\sqrt{\sum_{i=1}^n (\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2 + \langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2) + \langle T, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2}} \quad i = 1, \dots, n, \\ \langle \nu, T \rangle &= \frac{\langle T, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}}{\sqrt{\sum_{i=1}^n (\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2 + \langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2) + \langle T, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2}} =: \nu_T. \end{aligned}$$

In the above formulas, \mathbf{n}_{Eu} denotes the Euclidean unit normal along S and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{2n+1}}$ is the standard metric on \mathbb{R}^{2n+1} .

The Riemannian measure $\sigma_{\mathcal{R}}^{2n}$ on a smooth hypersurface S can be defined as *contraction*⁸ of the top-dimensional volume form $\sigma_{\mathcal{R}}^{2n+1}$ by the unit normal ν along S , i.e. $\sigma_{\mathcal{R}}^{2n} \lrcorner S := (\nu \lrcorner \sigma_{\mathcal{R}}^{2n+1})|_S$.

We say that $p \in S$ is a *characteristic point* if $\dim H_p = \dim(H_p \cap T_p S)$. By definition, the *characteristic set* of S is the set of all characteristic points, i.e. $C_S := \{x \in S : \dim H_p = \dim(H_p \cap T_p S)\}$. It is worth noting that $p \in C_S$ if, and only if, $|\mathcal{P}_H \nu(p)| = 0$. Since $|\mathcal{P}_H \nu(p)|$ is continuous along S , it follows that C_S is a closed subset of S , in the relative topology. We also remark that characteristic points are few, since under the preceding assumptions, the $(Q-1)$ -dimensional Hausdorff measure of C_S vanishes, i.e. $\mathcal{H}_{CC}^{Q-1}(C_S) = 0$; see, for instance, [2], [26].

Remark 1.4. Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth hypersurface. A straightforward application of Frobenius Theorem about integrable distributions shows that the topological dimension of C_S is strictly less than $(n+1)$; see also [17]. For other results about the size of C_S in \mathbb{H}^n , see [2].

In the sequel we will need another measure on hypersurfaces, the so-called *H-perimeter measure*; see [15], [10, 11], [26], [28, 29], [33], [36], [37] and references therein.

Definition 1.5 (σ_H^{2n} -measure). Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^1 -smooth non-characteristic hypersurface and let ν be the unit normal vector along S . The normalized projection of ν onto H is called *unit H-normal along S* , i.e. $\nu_H := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}$. The *H-perimeter form* $\sigma_H^{2n} \in \bigwedge^{2n}(T^*S)$ is the contraction of the volume form $\sigma_{\mathcal{R}}^{2n+1}$ of \mathbb{H}^n by the horizontal unit normal ν_H , i.e. $\sigma_H^{2n} \lrcorner S := (\nu_H \lrcorner \sigma_{\mathcal{R}}^{2n+1})|_S$.

If $C_S \neq \emptyset$ we extend σ_H^{2n} up to C_S by setting $\sigma_H^{2n} \lrcorner C_S = 0$. Moreover, it turns out that

$$\sigma_H^{2n} \lrcorner S = |\mathcal{P}_H \nu| \sigma_{\mathcal{R}}^{2n} \lrcorner S.$$

At each point $p \in S \setminus C_S$ one has $H_p = \text{span}_{\mathbb{R}}\{\nu_H(p)\} \oplus H_p S$, where $H_p S := H_p \cap T_p S$. So we can define, in the obvious way, the associated subbundles $HS \subset TS$ and $\nu_H S$ called, respectively, *horizontal tangent bundle* and *horizontal normal bundle* along S .

Remark 1.6. We have $\dim H_p S = \dim H - 1 = 2n - 1$ at each point $p \in S \setminus C_S$. Note that the definition of HS makes sense even if $p \in C_S$, but in such a case $\dim H_p S = \dim H_p = 2n$.

With respect to the horizontal o.n. frame $\mathcal{F}_H := \{X_1, Y_1, \dots, X_n, Y_n\}$, the unit H -normal ν_H can be written out as $\nu_H = \sum_{i=1}^n (\langle \nu_H, X_i \rangle X_i + \langle \nu_H, Y_i \rangle Y_i)$, where

$$\begin{aligned} \langle \nu_H, X_i \rangle &:= \frac{\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}}{\sqrt{\sum_{i=1}^n (\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2 + \langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2)}} \quad i = 1, \dots, n, \\ \langle \nu_H, Y_i \rangle &:= \frac{\langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}}{\sqrt{\sum_{i=1}^n (\langle X_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2 + \langle Y_i, \mathbf{n}_{\text{Eu}} \rangle_{\mathbb{R}^{2n+1}}^2)}} \quad i = 1, \dots, n. \end{aligned}$$

⁸Let M be a Riemannian manifold. The linear map $\lrcorner : \Lambda^r(T^*M) \rightarrow \Lambda^{r-1}(T^*M)$ is defined, for $X \in TM$ and $\omega^r \in \Lambda^r(T^*M)$, by $(X \lrcorner \omega^r)(Y_1, \dots, Y_{r-1}) := \omega^r(X, Y_1, \dots, Y_{r-1})$; see, for instance, [13]. This operation is called *contraction* or *interior product*.

Another important geometric object (see [28, 29], [11]) is given by $\varpi := \frac{\nu_T}{|\mathcal{P}_H \nu|}$. Clearly, ϖ is not defined at C_S , but one has $\varpi \in L_{loc}^1(S, \sigma_H^{2n})$.

Notation 1.7. We set $z^\perp := -C_H^{2n+1} z = (-y_1, x_1, \dots, -y_n, x_n) \in \mathbb{R}^{2n}$ and $X^\perp := -C_H^{2n+1} X$, for every $X \in H$. In particular, the horizontal tangent vector field ν_H^\perp is called characteristic direction along S . We also set $C_H(\varpi) := \varpi C_H^{2n+1}$. Note that $C_H(\varpi)\nu_H = -\varpi\nu_H^\perp = -\varpi \sum_{i=1}^n (-\langle \nu_H, Y_i \rangle X_i + \langle \nu_H, X_i \rangle Y_i)$.

Definition 1.8. Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth non-characteristic hypersurface. We call adapted frame along S any o.n. frame $\mathcal{F} := \{\tau_1, \dots, \tau_{2n+1}\}$ for $T\mathbb{H}^n$ such that:

$$(i) \tau_1|_S = \nu_H, \quad (ii) H_p S = \text{span}_{\mathbb{R}}\{\tau_2(p), \dots, \tau_{2n}(p)\} \text{ for every } p \in S, \quad (iii) \tau_{2n+1} := T.$$

We also set $I_H := \{1, 2, 3, \dots, 2n\}$ and $I_{HS} := \{2, 3, \dots, 2n\}$.

Lemma 1.9. Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth non-characteristic hypersurface and $p \in S$. Then, we can always choose an adapted o.n. frame $\mathcal{F} = \{\tau_1, \dots, \tau_{2n+1}\}$ along S such that $\langle \nabla_X \tau_i, \tau_j \rangle = 0$ at p for every $i, j \in I_{HS}$ and every $X \in H_p S$.

For a proof, see Lemma 3.8 in [29]. The next definitions can be found in [29], for k -step Carnot groups. Later on, unless otherwise specified, we shall assume that $S \subset \mathbb{H}^n$ is a \mathbf{C}^2 -smooth non-characteristic hypersurface.

Definition 1.10. Let $i = 1, 2$. We shall denote by $\mathbf{C}_{HS}^i(S)$ the space of functions whose i -th HS-derivatives are continuous on S . Analogously, for any open subset $\mathcal{U} \subseteq S$, we set $\mathbf{C}_{HS}^i(\mathcal{U})$, to denote the space of functions whose i -th HS-derivatives are continuous in \mathcal{U} . In particular, since C_S is closed, we may take $\mathcal{U} = S \setminus C_S$.

Warning 1.11. The last definition extends to the case $C_S \neq \emptyset$ by requiring that all i -th HS-derivatives be continuous at each characteristic point $p \in C_S$; see also Remark 1.6.

Let ∇^{TS} denote the connection on S induced from the Levi-Civita connection ∇ on \mathbb{H}^n . We define a partial connection ∇^{HS} associated with the subbundle $HS \subset TS$ by setting⁹

$$\nabla_X^{HS} Y := \mathcal{P}_{HS} (\nabla_X^{TS} Y) \quad \text{for every } X, Y \in \mathfrak{X}_{HS}^1 := \mathbf{C}^1(S, HS)$$

Starting from the orthogonal decomposition $H = \nu_H S \oplus HS$, it can be shown that

$$\nabla_X^{HS} Y = \nabla_X^H Y - \langle \nabla_X^H Y, \nu_H \rangle \nu_H \quad \text{for every } X, Y \in \mathfrak{X}_{HS}^1.$$

Definition 1.12. Given $\psi \in \mathbf{C}_{HS}^1(S)$, we define the HS-gradient of ψ to be the unique horizontal tangent vector field $\text{grad}_{HS} \psi \in \mathfrak{X}_{HS}^0 := \mathbf{C}(S, HS)$ such that $\langle \text{grad}_{HS} \psi, X \rangle = d\psi(X) = X\psi$ for every $X \in HS$. The HS-divergence $\text{div}_{HS} X$ of any $X \in \mathfrak{X}_{HS}^1$ is defined, at each point $p \in S$, by

$$\text{div}_{HS} X(p) := \text{Trace}(Y \longrightarrow \nabla_Y^{HS} X)(p) \quad (Y \in H_p S)$$

and one has $\text{div}_{HS} X \in \mathbf{C}(S)$. The HS-Laplacian Δ_{HS} is the 2nd order differential operator given by

$$\Delta_{HS} \psi := \text{div}_{HS} (\text{grad}_{HS} \psi) \quad \text{for every } \psi \in \mathbf{C}_{HS}^2(S).$$

Definition 1.13. The horizontal 2nd fundamental form of S , $B_H : \mathfrak{X}_{HS}^1 \times \mathfrak{X}_{HS}^1 \longrightarrow \mathbf{C}(S)$, is the bilinear map given by $B_H(X, Y) := \langle \nabla_X^H Y, \nu_H \rangle$ for every $X, Y \in \mathfrak{X}_{HS}^1$. The horizontal mean curvature is the trace of B_H , i.e. $\mathcal{H}_H := \text{Tr} B_H$. The torsion \mathbf{T}_{HS} of ∇^{HS} is defined by $\mathbf{T}_{HS}(X, Y) := \nabla_X^{HS} Y - \nabla_Y^{HS} X - \mathcal{P}_H[X, Y]$ for every $X, Y \in \mathfrak{X}_{HS}^1$.

If $n = 1$ the torsion is zero. Nevertheless, if $n > 1$ the torsion does not vanish, in general, because B_H is not symmetric; see [29].

Notation 1.14. Let $X \in \mathfrak{X}_H$ and let $\phi \in \mathbf{C}^2(\mathbb{H}^n)$. We shall denote by $X(\phi) := \langle \text{grad}_H \phi, X \rangle$ and $X^{(2)}(\phi) := X(X(\phi))$, the 1st and 2nd derivatives of ϕ along X , respectively. Furthermore, functions defined on S will be thought of as restrictions $\phi|_S$ to S of functions defined on \mathbb{H}^n .

Now we prove some identities useful for the sequel.

⁹ $\mathcal{P}_{HS} : TS \longrightarrow HS$ denotes the orthogonal projection operator of TS onto HS .

Lemma 1.15. *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth non-characteristic hypersurface and let $\phi \in \mathbf{C}^\infty(\mathbb{H}^n)$. Then*

$$\Delta_{HS} \phi = \Delta_H \phi + \mathcal{H}_H \frac{\partial \phi}{\partial \nu_H} - \langle \text{Hess}_H \phi \nu_H, \nu_H \rangle. \quad (2)$$

Proof. Using an adapted frame \mathcal{F} , we compute

$$\begin{aligned} \Delta_H \phi &= \sum_{i \in I_H} \left(\tau_i^{(2)} - \nabla_{\tau_i}^H \tau_i \right) (\phi) \\ &= \tau_1^{(2)}(\phi) - (\nabla_{\tau_1}^H \tau_1) (\phi) + \sum_{i \in I_{HS}} \left(\left(\tau_i^{(2)} - \nabla_{\tau_i}^{HS} \tau_i \right) (\phi) - \langle \nabla_{\tau_i}^H \tau_i, \tau_1 \rangle \tau_1(\phi) \right) \\ &= \tau_1^{(2)}(\phi) - (\nabla_{\tau_1}^H \tau_1) (\phi) + \Delta_{HS} \phi - \mathcal{H}_H \tau_1(\phi). \end{aligned}$$

Note that the first and the last identities come from the usual invariant definition of the Laplace operator on Riemannian manifolds (or vector bundles); see [20]. Now we claim that

$$\tau_1^{(2)}(\phi) - (\nabla_{\tau_1}^H \tau_1) (\phi) = \langle \text{Hess}_H \phi \tau_1, \tau_1 \rangle.$$

Assuming $\tau_1 = \sum_{i \in I_H} A_i^1 X_i$ yields

$$\tau_1^{(2)}(\phi) = \sum_{i \in I_H} \tau_1(A_i^1 X_i(\phi)) = \sum_{i, j \in I_H} (\tau_1(A_i^1) X_j(\phi) + A_i^1 A_j^1 X_j(X_i(\phi))).$$

Since

$$\nabla_{\tau_1}^H \tau_1 = \sum_{i, j \in I_H} \left(\tau_1(A_i^1) X_j + A_i^1 A_j^1 \underbrace{\nabla_{X_i}^H X_j}_{=0} \right),$$

the claim follows because

$$\tau_1^{(2)}(\phi) - (\nabla_{\tau_1}^H \tau_1) (\phi) = \sum_{i, j \in I_H} A_i^1 A_j^1 X_j(X_i(\phi)) = \langle \text{Hess}_H \phi \tau_1, \tau_1 \rangle. \quad (3)$$

□

Lemma 1.16. *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth non-characteristic hypersurface. Then*

$$\nabla_{\nu_H}^H \nu_H = -\frac{\text{grad}_{HS} \varpi}{\varpi} + \varpi \nu_H^\perp.$$

Proof. We make use of an adapted frame \mathcal{F} and identity (ii) of Lemma 3.12 in [29]. In the case of \mathbb{H}^n this identity can be rewritten as follows:

$$\left\langle \nabla_{\tau_{2n+1}^{TS}}^H \nu_H, \tau_j \right\rangle = \tau_j(\varpi) + \frac{1}{2} \langle C_H^{2n+1} \nu_H, \tau_j \rangle - \langle C_{\tau_{2n+1}}^{TS}, \tau_j \rangle \quad \text{for every } j \in I_{HS},$$

where $\tau_{2n+1}^{TS} := \tau_{2n+1} - \varpi \tau_1 = T - \varpi \nu_H$. Therefore

$$\langle \nabla_T^H \nu_H, \tau_j \rangle - \varpi \langle \nabla_{\nu_H}^H \nu_H, \tau_j \rangle = \tau_j(\varpi) + \frac{1}{2} \langle C_H^{2n+1} \nu_H, \tau_j \rangle + \varpi^2 \langle C_H^{2n+1} \nu_H, \tau_j \rangle \quad \text{for every } j \in I_{HS}.$$

By using (iii) of Lemma 3.13 in [29], we also get that $\langle \nabla_T^H \nu_H, \tau_j \rangle = \frac{1}{2} \langle C_H^{2n+1} \nu_H, \tau_j \rangle$ and the thesis easily follows. □

Note that, as a byproduct of identity (3) and Lemma 1.16, we immediately get the next:

Lemma 1.17. *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth non-characteristic hypersurface. Then*

$$\langle \text{Hess}_H \phi \nu_H, \nu_H \rangle = \frac{\partial^2 \phi}{\partial \nu_H^2} + \left\langle \frac{\text{grad}_{HS} \varpi}{\varpi}, \text{grad}_{HS} \phi \right\rangle - \varpi \frac{\partial \phi}{\partial \nu_H^\perp}$$

for every $\phi \in \mathbf{C}^\infty(\mathbb{H}^n)$.

1.3 Horizontal integration by parts

First, let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth non-characteristic hypersurface.

Definition 1.18 (Horizontal tangential operators). *Let $\mathcal{D}_{HS} : \mathfrak{X}_{HS}^1 \longrightarrow \mathbf{C}(S)$ be the 1st order differential operator given by*

$$\mathcal{D}_{HS}(X) := \operatorname{div}_{HS} X + \varpi \langle C_H^{2n+1} \nu_H, X \rangle = \operatorname{div}_{HS} X - \varpi \langle \nu_H^\perp, X \rangle \quad \text{for every } X \in \mathfrak{X}_{HS}^1.$$

Furthermore, let $\mathcal{L}_{HS} : \mathbf{C}_{HS}^2(S) \longrightarrow \mathbf{C}(S)$ be the 2nd order differential operator defined as

$$\mathcal{L}_{HS} \varphi := \Delta_{HS} \varphi - \varpi \frac{\partial \varphi}{\partial \nu_H^\perp} \quad \text{for every } \varphi \in \mathbf{C}_{HS}^2(S).$$

Note that $\mathcal{D}_{HS}(\varphi X) = \varphi \mathcal{D}_{HS} X + \langle \operatorname{grad}_{HS} \varphi, X \rangle$ for every $X \in \mathfrak{X}_{HS}^1$ and every $\varphi \in \mathbf{C}_{HS}^1(S)$. Moreover $\mathcal{L}_{HS} \varphi = \mathcal{D}_{HS}(\operatorname{grad}_{HS} \varphi)$ for every $\varphi \in \mathbf{C}_{HS}^2(S)$.

The previous definition is motivated by Theorem 3.17 in [29]. For the sake of simplicity, let us illustrate the case of the 1st Heisenberg group \mathbb{H}^1 .

Remark 1.19. *Let $S \subset \mathbb{H}^1$ be a smooth non-characteristic surface. The H -perimeter form σ_H^2 is given by $\sigma_H^2 \lrcorner S = (\nu_H^\perp)^* \wedge \theta|_S$, where $(\nu_H^\perp)^* := -\nu_{H_Y} dx + \nu_{H_X} dy$. Note that since HS is 1-dimensional, ν_H^\perp turns out to be the unique horizontal tangent direction along S . Now let us compute the exterior derivative of the 1-form $(X \lrcorner \sigma_H^2)|_S$, for any $X \in \mathfrak{X}_{HS}^1$. Assuming $X = f\nu_H^\perp$, for some smooth $f : S \longrightarrow \mathbb{R}$, yields*

$$d(X \lrcorner \sigma_H^2)|_S = d(X \lrcorner ((\nu_H^\perp)^* \wedge \theta))|_S = d(f\theta)|_S = (df \wedge \theta + f d\theta)|_S.$$

Since $d\theta = -dx \wedge dy$, by means of a linear change of variables, we immediately get that

$$d\theta|_S = -(\nu_H)^* \wedge (\nu_H^\perp)^*|_S = -\varpi \sigma_H^2|_S.$$

Finally, since $(\nu_H)^* \wedge \theta|_S = 0$, we obtain

$$d(X \lrcorner \sigma_H^2)|_S = (\nu_H^\perp f - f\varpi) \sigma_H^2|_S = \mathcal{D}_{HS}(X) \sigma_H^2|_S.$$

Now let ∂S be a (piecewise) $(2n-1)$ -dimensional \mathbf{C}^1 -smooth manifold, oriented by its unit normal vector $\eta \in TS$ and let $\sigma_{\mathcal{R}}^{2n-1}$ be the Riemannian measure on ∂S , i.e. $\sigma_{\mathcal{R}}^{2n-1} \lrcorner \partial S = (\eta \lrcorner \sigma_{\mathcal{R}}^{2n})|_{\partial S}$. If $X \in \mathbf{C}(S, TS)$, then $(X \lrcorner \sigma_H^{2n})|_{\partial S} = \langle X, \eta \rangle |\mathcal{P}_H \nu| \sigma_{\mathcal{R}}^{2n-1} \lrcorner \partial S$. The characteristic set $C_{\partial S}$ of ∂S is defined as $C_{\partial S} := \{p \in \partial S : |\mathcal{P}_H \nu| |\mathcal{P}_{HS} \eta| = 0\}$. The unit HS -normal along ∂S is given by $\eta_{HS} := \frac{\mathcal{P}_{HS} \eta}{|\mathcal{P}_{HS} \eta|}$ and we may define a homogeneous measure σ_{HS}^{2n-1} along ∂S by setting $\sigma_{HS}^{2n-1} \lrcorner \partial S := (\eta_{HS} \lrcorner \sigma_H^{2n})|_{\partial S}$. As for the H -perimeter, the measure σ_{HS}^{2n-1} can be represented by means of the Riemannian measure $\sigma_{\mathcal{R}}^{2n-1}$ and it turns out that $\sigma_{HS}^{2n-1} \lrcorner \partial S = |\mathcal{P}_H \nu| |\mathcal{P}_{HS} \eta| \sigma_{\mathcal{R}}^{2n-1} \lrcorner \partial S$.

The following horizontal integration by parts formula can be found in [29].

Theorem 1.20. *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth compact non-characteristic hypersurface with piecewise \mathbf{C}^1 -smooth boundary ∂S . Then*

$$\int_S \mathcal{D}_{HS}(X) \sigma_H^{2n} = - \int_S \mathcal{H}_H \langle X, \nu_H \rangle \sigma_H^{2n} + \int_{\partial S} \langle X, \eta_{HS} \rangle \sigma_{HS}^{2n-1} \quad \text{for every } X \in \mathfrak{X}_H^1.$$

Remark 1.21. *Let $S \subset \mathbb{H}^n$ be \mathbf{C}^2 -smooth compact hypersurface with piecewise \mathbf{C}^1 -smooth boundary ∂S . As already said, in this case it turns out that $\dim_{\text{Eu-Hau}}(\operatorname{Car}_S) \leq n$. Just for the case $n = 1$, we will further suppose that C_S is contained in a finite union of \mathbf{C}^1 -smooth horizontal curves. Under these assumptions, one can show that there exists a family $\{\mathcal{U}_\epsilon\}_{\epsilon \geq 0}$ of open subsets of S with piecewise \mathbf{C}^1 -smooth boundaries such that:*

- (i) $\operatorname{Car}_S \subseteq \mathcal{U}_\epsilon$ for every $\epsilon > 0$;
- (ii) $\sigma_{\mathcal{R}}^{2n}(\mathcal{U}_\epsilon) \longrightarrow 0$ for $\epsilon \rightarrow 0^+$;
- (iii) $\int_{\partial \mathcal{U}_\epsilon} |\mathcal{P}_H \nu| \sigma_{\mathcal{R}}^{2n-1} \longrightarrow 0$ for $\epsilon \rightarrow 0^+$.

By using (iii) above, we infer that $\sigma_{HS}^{2n-1}(\partial\mathcal{U}_\epsilon) \rightarrow 0$ for $\epsilon \rightarrow 0^+$. Therefore, in order to extend Theorem 1.20 to the case $C_S \neq \emptyset$, it is then sufficient to apply it to the non-characteristic hypersurface $S_\epsilon := S \setminus \mathcal{U}_\epsilon$. We have

$$\int_{S_\epsilon} \mathcal{D}_{HS}(X) \sigma_H^{2n} = - \int_{S_\epsilon} \mathcal{H}_H \langle X, \nu_H \rangle \sigma_H^{2n} + \int_{\partial S} \langle X, \eta_{HS} \rangle \sigma_{HS}^{2n-1} - \int_{\partial\mathcal{U}_\epsilon} \langle X, \eta_{HS} \rangle \sigma_{HS}^{2n-1}. \quad (4)$$

Note that $\operatorname{div}_{HS} X = \operatorname{div}_H X - \langle \mathcal{J}_H X, \nu_H \rangle \in \mathbf{C}(S \setminus C_S) \cap L^\infty(S)$. Furthermore, since $\varpi \in L^1(S, \sigma_H^{2n})$, by using dominated convergence and (ii), we get that

$$\lim_{\epsilon \rightarrow 0^+} \int_{S_\epsilon} (\operatorname{div}_{HS} X + \langle C_H(\varpi) \nu_H, X \rangle) \sigma_H^{2n} = \int_S (\operatorname{div}_{HS} X + \langle C_H(\varpi) \nu_H, X \rangle) \sigma_H^{2n}.$$

It is not difficult¹⁰ to show that $\mathcal{H}_H \in L^1(S, \sigma_H^{2n})$. Therefore

$$\int_S \mathcal{H}_H \langle X, \nu_H \rangle \sigma_H^{2n} = \lim_{\epsilon \rightarrow 0^+} \int_{S_\epsilon} \mathcal{H}_H \langle X, \nu_H \rangle \sigma_H^{2n}.$$

Finally, by applying (iii), we see that the integral over $\partial\mathcal{U}_\epsilon$ in (4) goes to 0 as long as $\epsilon \rightarrow 0^+$. This shows the validity of Theorem 1.20 even if $C_S \neq \emptyset$. An analogous argument can be used, in order to prove the validity of some Green-type formulas up to C_S , at least if we consider functions belonging to $\mathbf{C}_{HS}^2(S)$.

We now collect some useful formula concerning the operator \mathcal{L}_{HS} .

Theorem 1.22. *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth compact hypersurface with piecewise \mathbf{C}^1 -smooth boundary ∂S . If $n = 1$, assume further that C_S is contained in a finite union of \mathbf{C}^1 -smooth horizontal curves. Then, the following hold:*

- (i) $\int_S \mathcal{L}_{HS} \varphi \sigma_H^{2n} = 0$ for every compactly supported $\varphi \in \mathbf{C}_{HS}^2(S)$;
- (ii) $\int_S \mathcal{L}_{HS} \varphi \sigma_H^{2n} = \int_{\partial S} \partial\varphi / \partial\eta_{HS} \sigma_{HS}^{2n-1}$ for every $\varphi \in \mathbf{C}_{HS}^2(S)$;
- (iii) $\int_S \psi \mathcal{L}_{HS} \varphi \sigma_H^{2n} = \int_S \varphi \mathcal{L}_{HS} \psi \sigma_H^{2n}$ for every compactly supported $\varphi, \psi \in \mathbf{C}_{HS}^2(S)$;
- (iv) $\int_S (\psi \mathcal{L}_{HS} \varphi - \varphi \mathcal{L}_{HS} \psi) \sigma_H^{2n} = \int_{\partial S} (\psi \partial\varphi / \partial\eta_{HS} - \varphi \partial\psi / \partial\eta_{HS}) \sigma_{HS}^{2n-1}$ for every $\varphi, \psi \in \mathbf{C}_{HS}^2(S)$;
- (v) $\int_S \psi \mathcal{L}_{HS} \varphi \sigma_H^{2n} = - \int_S \langle \operatorname{grad}_{HS} \varphi, \operatorname{grad}_{HS} \psi \rangle \sigma_H^{2n} + \int_{\partial S} \psi \partial\varphi / \partial\eta_{HS} \sigma_{HS}^{2n-1}$ for every $\varphi, \psi \in \mathbf{C}_{HS}^2(S)$;
- (vi) $\int_S \mathcal{L}_{HS} (\varphi^2) \sigma_H^{2n} = 2 \int_S \varphi \mathcal{L}_{HS} \varphi \sigma_H^{2n} + 2 \int_S |\operatorname{grad}_{HS} \varphi|^2 \sigma_H^{2n} = \int_{\partial S} \partial\varphi^2 / \partial\eta_{HS} \sigma_{HS}^{2n-1}$ for every $\varphi \in \mathbf{C}_{HS}^2(S)$.

Proof. See [29] for the non-characteristic case. If $C_S \neq \emptyset$, the proof follows by arguing exactly as in Remark 1.21. \square

We end this section by stating a first consequence.

Proposition 1.23 (Hopf's Lemma). *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth closed hypersurface. If $n = 1$, assume further that C_S is contained in a finite union of \mathbf{C}^1 -smooth horizontal curves. If φ is a \mathbf{C}_{HS}^2 -smooth function such that $\mathcal{L}_{HS} \varphi \geq 0$ everywhere in $S \setminus C_S$, then φ is a constant function and $\mathcal{L}_{HS} \varphi = 0$.*

Proof. Since $\partial S = \emptyset$, by using (iii) in Theorem 1.22, we have $\int_S \mathcal{L}_{HS} \varphi = 0$ and since $\mathcal{L}_{HS} \varphi \geq 0$, it follows that $\mathcal{L}_{HS} \varphi = 0$ on $S \setminus C_S$. Thus we get that $\int_S |\operatorname{grad}_{HS} \varphi|^2 \sigma_H^{2n} = 0$ and hence $|\operatorname{grad}_{HS} \varphi| = 0$. Now we claim that φ must be constant. If $n > 1$, this follows by the *bracket-generating condition* (i.e. $[HS, HS] = TS$) which is satisfied by the horizontal subbundle HS of any smooth non-characteristic hypersurface $S \subset \mathbb{H}^n$. Note that under our assumptions, $S \setminus C_S$ turns out to be a finite union of non-characteristic open and connected subsets of S . Hence, the claim follows by the continuity of φ . If $n = 1$, arguing as above implies that φ is constant along each leaf of the so-called characteristic foliation of S and the thesis easily follows. \square

¹⁰Indeed, note that $|\mathcal{H}_H| = \frac{|\operatorname{div}_H(\mathcal{P}_H \nu) - \langle \operatorname{grad}_H |\mathcal{P}_H \nu|, \nu_H \rangle|}{|\mathcal{P}_H \nu|}$.

1.4 Further remarks about the horizontal integration by parts up to C_S

It is well-known that Stokes formula is concerned with integrating a k -form over a k -dimensional manifold with boundary. A common way to state this fundamental result is the following:

Proposition 1.24 (Stokes formula). *Let M be an oriented k -dimensional manifold of class \mathbf{C}^2 with boundary ∂M . Then*

$$\int_M d\alpha = \int_{\partial M} \alpha$$

for every compactly supported $(k-1)$ -form α of class \mathbf{C}^1 .

One requires M to be of class \mathbf{C}^2 for a technical reason concerning “pull-back” of differential forms. Without much effort, it is possible to extend Proposition 1.24 to the case where:

(\star) *M is of class \mathbf{C}^1 and α is a $(k-1)$ -form such that α and $d\alpha$ are continuous.*

For a more detailed discussion we refer the reader to the book by Taylor [42]. Nevertheless, it is worth noting that much more general versions of Stokes formula are available in literature (see, for instance, [13]) and that researches aiming to generalize it are still intense.

In our setting, we have already discussed horizontal integration by parts formulas. However, the previous condition (\star) can easily be used in order to extend our formulas to vector fields (and functions) possibly singular at the characteristic set C_S . More precisely, let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth hypersurface with piecewise \mathbf{C}^1 -smooth boundary ∂S and let $X \in \mathbf{C}^1(S \setminus C_S, HS)$. Furthermore, set

$$\alpha_X := (X \lrcorner \sigma_H^{2n})|_S.$$

Then, condition (\star) simply requires that α_X and $d\alpha_X$ be continuous on S . Note that X is of class \mathbf{C}^1 out of C_S but may be singular at C_S .

Definition 1.25. *Let $X \in \mathbf{C}^1(S \setminus C_S, HS)$ and set $\alpha_X := (X \lrcorner \sigma_H^{2n})|_S$. We say that X is admissible (for the horizontal divergence formula) if, and only if, the differential forms α_X and $d\alpha_X$ are continuous on all of S . Furthermore, we say that $\phi \in \mathbf{C}_{HS}^2(S \setminus C_S)$ is admissible if, and only if, $\text{grad}_{HS} \phi$ is admissible (for the horizontal divergence formula).*

In other words, we have the formulas:

- $\int_S \mathcal{D}_{HS}(X) \sigma_H^{2n} = \int_{\partial S} \langle X, \eta_{HS} \rangle \sigma_{HS}^{2n-1}$ for every admissible $X \in \mathbf{C}^1(S \setminus C_S, HS)$;
- $\int_S \mathcal{L}_{HS} \phi \sigma_H^{2n} = \int_{\partial S} \langle \text{grad}_{HS} \phi, \eta_{HS} \rangle \sigma_{HS}^{2n-1}$ for every admissible $\phi \in \mathbf{C}_{HS}^2(S \setminus C_S, HS)$.

For later purposes, we have now to define a space of “admissible” functions for the horizontal Green’s formulas (iii)-(vi) of Theorem 1.22.

Definition 1.26. *We say that $\phi \in \mathbf{C}_{HS}^2(S \setminus C_S)$ is admissible for the horizontal Green’s formulas if, and only if, $\psi \text{grad}_{HS} \phi$ is admissible (for the horizontal divergence formula) for every $\psi \in \mathbf{C}_{HS}^2(S \setminus C_S)$ such that $\psi \text{grad}_{HS} \psi$ is admissible (for the horizontal divergence formula). We shall denote by $\Phi(S)$ the space of all admissible functions for the horizontal Green’s formulas.*

Obviously, the formulas (iii)-(vi) in Theorem 1.22 extend to functions belonging to the space $\Phi(S)$ of all admissible functions for the horizontal Green’s formulas.

1.5 Closed eigenvalue problem for \mathcal{L}_{HS}

Let $S \subset \mathbb{H}^n$ be a compact closed \mathbf{C}^2 -smooth hypersurface and denote by $L^2(S, \sigma_H^{2n})$ the space of all σ_H^{2n} -measurable functions f on S such that $\int_S |f|^2 \sigma_H^{2n} < +\infty$. Obviously, $L^2(S, \sigma_H^{2n})$ is a Hilbert space with respect to the inner product (\cdot, \cdot) defined by $(f, g) := \int_S f g \sigma_H^{2n}$ for every $f, g \in L^2(S, \sigma_H^{2n})$. Its associated norm is denoted by $\|\cdot\|$. For σ_H^{2n} -measurable horizontal tangent vector fields we define an inner product (with associated norm $\|\cdot\|_0$) by setting

$$(X, Y)_0 := \int_S \langle X, Y \rangle \sigma_H^{2n} \quad \left(\|X\|_0^2 := \int_S |X|^2 \sigma_H^{2n} \right). \quad (5)$$

The resulting metric space, denoted by $\mathcal{L}^2(S, \sigma_H^{2n})$, identifies with the space of σ_H^{2n} -measurable horizontal tangent vector fields for which the integral (5) is finite. If $f \in \mathbf{C}_{HS}^1(S)$ and $X \in \mathfrak{X}_{HS}^1 = \mathbf{C}^1(S, HS)$, then

$$(\text{grad}_{HS} f, X)_0 = -(f, \mathcal{D}_{HS} X).$$

However, we would like to include in our analysis, more general functions.

Definition 1.27. Given a function $f \in L^2(S, \sigma_H^{2n})$, we say that $Y \in \mathcal{L}^2(S, \sigma_H^{2n})$ is a HS-weak derivative of f if, and only if, one has $(Y, X)_0 = -\int_S f \mathcal{D}_{HS} X \sigma_H^{2n}$ for every $X \in \mathfrak{X}_{HS}^1$.

Since it is not difficult to show that there exists at most one such $Y \in \mathcal{L}^2(S, \sigma_H^{2n})$, we shall write $Y = \text{grad}_{HS} f$. In the sequel, we will denote by $\mathcal{H}(S, \sigma_H^{2n})$ the subspace of $L^2(S, \sigma_H^{2n})$ of functions having HS-weak derivatives. This space is endowed with the inner product $(f, g)_1 := (f, g) + (\text{grad}_{HS} f, \text{grad}_{HS} g)_0$ with associated norm $\|f\|_1^2 := \|f\|^2 + \|\text{grad}_{HS} f\|_0^2$.

Remark 1.28. We stress that the space $\Phi(S)$ of all admissible functions for the horizontal Green's formulas is a subspace of $\mathcal{H}(S, \sigma_H^{2n})$.

The energy integral in $\mathcal{H}(S, \sigma_H^{2n})$ is the symmetric bilinear form defined by

$$\mathcal{E}(f, g) := \int_S \langle \text{grad}_{HS} f, \text{grad}_{HS} g \rangle \sigma_H^{2n}$$

for every $f, g \in \mathcal{H}(S, \sigma_H^{2n})$.

Here we are concerned with the validity, under suitable assumptions, of the formula:

$$(\mathcal{L}_{HS} \phi, f) = -\mathcal{E}(\phi, f). \quad (6)$$

By the results of Section 1.3, this formula holds true for every $\phi \in \mathbf{C}_{HS}^2(S)$ and for every $f \in \mathbf{C}_{HS}^1(S)$. More generally, let us assume $\phi \in \Phi(S)$; see Definition 1.26 in Section 1.4. In this case formula (6) defines a linear functional L_ϕ on $\mathbf{C}_{HS}^1(S)$ as a subspace of $\mathcal{H}(S, \sigma_H^{2n})$ satisfying:

$$|L_\phi(f)| \leq \|\text{grad}_{HS} \phi\|_0 \|\text{grad}_{HS} f\|_0 \leq \|\text{grad}_{HS} \phi\|_0 \|f\|_1.$$

In other words, L_ϕ turns out to be a bounded linear functional on $\mathbf{C}_{HS}^1(S) \subset \mathcal{H}(S, \sigma_H^{2n})$ with norm bounded from the above by $\|\text{grad}_{HS} \phi\|_0$ and so it can be extended to a bounded linear functional on all of $\mathcal{H}(S, \sigma_H^{2n})$. Therefore, formula (6) turns out to be valid for every $\phi \in \Phi(S)$ and every $f \in \mathcal{H}(S, \sigma_H^{2n})$.

We are now in a position to formulate the closed eigenvalue problem for the operator \mathcal{L}_{HS} .

Problem 1.29. Let $S \subset \mathbb{H}^n$ be a compact closed hypersurface of class \mathbf{C}^2 . The closed eigenvalue problem for the operator \mathcal{L}_{HS} on S is to find all real numbers λ for which there exist non-trivial solutions $\varphi \in \Phi(S)$ to the following

$$(P) \quad \begin{cases} \mathcal{L}_{HS} \varphi = -\lambda \varphi & \text{on } S \setminus C_S \\ \int_S \varphi \sigma_H^{2n} = 0. \end{cases}.$$

Some remarks are in order.

- If $\phi \in \Phi(S)$ is an eigenfunction of Problem 1.29, then its eigenvalue λ must be non-negative. The proof of this claim is based on the identity $\int_S (\phi \mathcal{L}_{HS} \phi + |\text{grad}_{HS} \phi|^2) \sigma_H^{2n} = 0$, from which we get

$$\int_S |\text{grad}_{HS} \phi|^2 \sigma_H^{2n} = - \int_S \phi \mathcal{L}_{HS} \phi \sigma_H^{2n} = \lambda \int_S \phi^2 \sigma_H^{2n}.$$

Note that if $\lambda = 0$, it follows that ϕ must be constant along S ; see Proposition 1.23. Hence $\lambda_0 = 0$ cannot be an eigenvalue of Problem 1.29.

- Eigenspaces belonging to different eigenvalues are orthogonal in $L^2(S, \sigma_H^{2n})$. This follows basically from formula (iii) in Theorem 1.22, i.e.

$$\int_S (\phi \mathcal{L}_{HS} \psi - \psi \mathcal{L}_{HS} \phi) \sigma_H^{2n} = 0.$$

Indeed, let ϕ, ψ be eigenfunctions of λ and τ , respectively. Then

$$\int_S (\phi \mathcal{L}_{HS} \psi - \psi \mathcal{L}_{HS} \phi) \sigma_H^{2n} = (\lambda - \tau) \int_S \phi \psi \sigma_H^{2n} = 0.$$

- The dimension of each eigenspace is called the *multiplicity* of the eigenvalue. Later on, we shall list the eigenvalues as $0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow +\infty$, with each eigenvalue repeated according its multiplicity.

Theorem 1.30 (Rayleigh's Theorem). *Let $S \subset \mathbb{H}^n$ be a \mathbf{C}^2 -smooth compact closed hypersurface and let us consider the closed eigenvalue problem for the operator \mathcal{L}_{HS} on S ; see Problem 1.29. Assume that there exist eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow +\infty, \quad (7)$$

where each eigenvalue is repeated the number of times equal to its multiplicity. Then, for every non-zero function $f \in \mathcal{H}(S, \sigma_H^{2n})$ one has $\lambda_1 \leq \frac{\mathcal{E}(f, f)}{\|f\|^2}$, with equality if, and only if, f is an eigenfunction of λ_1 . If $\{\phi_1, \phi_2, \dots\}$ is a complete orthonormal basis of $L^2(S, \sigma_H^{2n})$ such that ϕ_j is an eigenfunction of λ_j for every $j = 1, 2, \dots$, then for any non-zero function $f \in \mathcal{H}(S, \sigma_H^{2n})$ such that $(f, \phi_1) = \dots = (f, \phi_{k-1}) = 0$, it turns out that $\lambda_k \leq \frac{\mathcal{E}(f, f)}{\|f\|^2}$, with equality if, and only if, f is an eigenfunction of λ_k .

Proof. The proof is based on the validity of formula (6) and it is completely analogous to the Riemannian one, for which we refer the reader to Chavel's book [7]. \square

Theorem 1.31 (Max-Min Theorem). *Let $\eta_1, \dots, \eta_{k-1} \in L^2(S, \sigma_H^{2n})$ and let $\mu = \inf \frac{\mathcal{E}(f, f)}{\|f\|^2}$, where f belongs to the subspace, less the origin, of functions in $\mathcal{H}(S, \sigma_H^{2n})$ orthogonal to $\eta_1, \dots, \eta_{k-1}$ in $L^2(S, \sigma_H^{2n})$. Then, for eigenvalues given in (7), we have $\mu \leq \lambda_k$. If $\eta_1, \dots, \eta_{k-1}$ are orthonormal and each η_l is an eigenfunction of λ_l for every $l = 1, \dots, k-1$, then $\mu = \lambda_k$.*

Proof. The proof can be done by repeating the arguments of the Riemannian case; see [7]. \square

2 Closed eigenvalue problem for \mathcal{L}_{HS} on Isoperimetric Profiles

2.1 Heisenberg Isoperimetric Profiles

This section is devoted to study some features of the unit Isoperimetric Profile $\mathbb{S}_{\mathbb{H}^n} \subset \mathbb{H}^n$. As already said, Isoperimetric Profiles are compact hypersurfaces, similar to ellipsoids, which turn out to be fibred by CC-geodesics; see also [21]. Their importance comes from a longstanding conjecture, usually attributed to Pansu, stating that they minimize the H -perimeter among finite H -perimeter sets having fixed volume. In other words, the Isoperimetric Profile is the main (perhaps only) candidate to solve the sub-Riemannian isoperimetric problem in \mathbb{H}^n . There is a wide literature on this subject and many partial answers, for which we refer the reader to [4], [10], [11], [12], [21], [22, 23], [24], [25], [32], [33, 34], [37]. In particular, in [37] it is shown that the conjecture is true for compact \mathbf{C}^2 -smooth surfaces in \mathbb{H}^4 .

Let us preliminary state some basics about CC-geodesics in \mathbb{H}^n . By definition, CC-geodesics are horizontal curves which minimize the CC-distance d_{CC} between two given points. They turn out to be solutions of the following system of O.D.E.s:

$$\begin{cases} \dot{x} = P_H \\ \dot{P}_H = -P_{2n+1} C_H^{2n+1} P_H \\ \dot{P}_{2n+1} = 0, \end{cases}$$

or equivalently, we have to solve $\dot{P}_H = P_{2n+1} P_H^\perp$, where $P_H = (P_1, \dots, P_{2n})^{\text{Tr}}$, $|P_H| = 1$ and P_{2n+1} is a constant parameter along the solution. We stress that $P = (P_H, P_{2n+1}) \in \mathbb{R}^{2n+1}$ can be thought of as Lagrangian multipliers and that the previous system can be deduced by directly minimizing a constrained Lagrangian; see [1] and references therein, or [31]. Unlike the Riemannian case, CC-geodesics in \mathbb{H}^n depend not only on the initial point $x(0)$ and on the initial direction $P_H(0)$, but also on the parameter P_{2n+1} which is a type of curvature for the solution. Indeed, if $P_{2n+1} = 0$, CC-geodesics are Euclidean horizontal lines, while if $P_{2n+1} \neq 0$, every CC-geodesic $x(t)$ starting from a fixed point x_0 is a sort of "helix"¹¹. In this case, the horizontal projection of $x(t)$ onto $H_0 = \mathbb{R}^{2n}$ belongs to a sphere whose radius only depends on P_{2n+1} . In particular, $x(t)$ touches the T -line through x_0 infinitely many times. However, $x(t)$ minimizes the CC-distance from x_0 only on the connected subset determined by the first point x_1 belonging to the T -line through x_0 . Set $x_0 = \mathcal{S}$, $x_1 = \mathcal{N}$, and call them the North and South poles. By rotating this curve around the T -line through x_0 , we obtain a closed convex hypersurface of constant horizontal mean curvature called *Isoperimetric Profile*. Below we will study some features of a "model" Isoperimetric Profile having barycenter at the identity $0 \in \mathbb{H}^n$. Of course, due to left-translations and dilations, this is not restrictive at all.

¹¹If $n = 1$, $x(t)$ is a circular helix with axis parallel to the vertical direction T and whose slope depends on P_3 . We stress that the projection of $x(t)$ onto \mathbb{R}^2 turns out to be a circle whose radius explicitly depends on P_3 .

Let $\rho := \|z\| = \sqrt{\sum_{i=1}^n (x_i^2 + y_i^2)}$ be the (Euclidean) norm of $z = (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$. Moreover, let $u_0 : B_1(0) := \{z \in \mathbb{R}^{2n} : 0 \leq \rho \leq 1\} \rightarrow \mathbb{R}$,

$$u_0(\rho) := \frac{\pi}{8} + \frac{\rho}{4} \sqrt{1 - \rho^2} - \frac{\rho}{4} \arcsin \rho \quad (0 \leq \rho \leq 1); \quad (8)$$

see also [10, 11, 12], [21]. Setting $\mathbb{S}_{\mathbb{H}^n}^\pm := \{p = \exp(\sum_{i=1}^n (x_i X_i + y_i Y_i) + tT) \in \mathbb{H}^n : t = \pm u_0\}$ we define the *Heisenberg unit Isoperimetric Profile* $\mathbb{S}_{\mathbb{H}^n}$ to be the compact hypersurface obtained by gluing together $\mathbb{S}_{\mathbb{H}^n}^+$ and $\mathbb{S}_{\mathbb{H}^n}^-$, i.e. $\mathbb{S}_{\mathbb{H}^n} = \mathbb{S}_{\mathbb{H}^n}^+ \cup \mathbb{S}_{\mathbb{H}^n}^-$. Since $\nabla_{\mathbb{R}^{2n}} u_0 = u_0'(\rho) \frac{z}{\rho}$, it follows that the Euclidean unit normal n_{Eu} along $\mathbb{S}_{\mathbb{H}^n}^\pm$ is given by $n_{\text{Eu}}^\pm = \frac{(-\nabla_{\mathbb{R}^{2n}} u_0, \pm 1)}{\sqrt{1 + \|\nabla_{\mathbb{R}^{2n}} u_0\|^2}}$, from which we compute the Riemannian unit normal ν . More precisely, we have

$$\nu^\pm = \frac{\left(-\nabla_{\mathbb{R}^{2n}} u_0 \pm \frac{z^\perp}{2}, \pm 1\right)}{\sqrt{1 + \|\nabla_{\mathbb{R}^{2n}} u_0\|^2 + \frac{\rho^2}{4}}}.$$

Now since

$$\frac{\partial u_0}{\partial \rho}(\rho) = \frac{1}{4} \left(\sqrt{1 - \rho^2} - \frac{\rho^2}{\sqrt{1 - \rho^2}} - \frac{1}{\sqrt{1 - \rho^2}} \right) = \frac{-\rho^2}{2\sqrt{1 - \rho^2}},$$

we get that

$$\nu_H^\pm = \frac{\left(-\nabla_{\mathbb{R}^{2n}} u_0 \pm \frac{z^\perp}{2}\right)}{\sqrt{\|\nabla_{\mathbb{R}^{2n}} u_0\|^2 + \frac{\rho^2}{4}}} = z \pm \frac{\sqrt{1 - \rho^2}}{\rho} z^\perp$$

and so

$$(\nu_H^\pm)^\perp = \left(-z \pm \frac{\sqrt{1 - \rho^2}}{\rho} z^\perp\right)^\perp = z^\perp \mp \frac{\sqrt{1 - \rho^2}}{\rho} z.$$

Moreover, since $|\mathcal{P}_H(\nu^\pm)| = \frac{\sqrt{\|\nabla_{\mathbb{R}^{2n}} u_0\|^2 + \frac{\rho^2}{4}}}{\sqrt{1 + \|\nabla_{\mathbb{R}^{2n}} u_0\|^2 + \frac{\rho^2}{4}}}$, we also get that

$$\sigma_H^{2n} \mathbb{L} \mathbb{S}_{\mathbb{H}^n}^\pm = |\mathcal{P}_H(\nu^\pm)| \sigma_{\mathbb{R}^n}^{2n} \mathbb{L} \mathbb{S}_{\mathbb{H}^n}^\pm = \sqrt{\|\nabla_{\mathbb{R}^{2n}} u_0\|^2 + \frac{\rho^2}{4}} dz \mathbb{L} B_1(0) = \frac{\rho}{2\sqrt{1 - \rho^2}} dz \mathbb{L} B_1(0).$$

Furthermore $\mathcal{H}_H = -\text{div}_H \nu_H = -2n$. By its own definition, we have

$$\varpi^\pm = \frac{\pm 1}{\|\nabla_{\mathbb{R}^{2n}} u_0 + \frac{z^\perp}{2}\|} = \frac{\pm 1}{\sqrt{\|\nabla_{\mathbb{R}^{2n}} u_0\|^2 + \frac{\rho^2}{4}}} = \frac{\pm 1}{\sqrt{\frac{\rho^4}{4(1 - \rho^2)} + \frac{\rho^2}{4}}} = \pm 2 \frac{\sqrt{1 - \rho^2}}{\rho}.$$

Note that $\text{grad}_H \varphi \equiv \nabla_{\mathbb{R}^{2n}} \varphi$ for every function $\varphi : \mathbb{H}^n \rightarrow \mathbb{R}$ independent of t . So we get

$$\text{grad}_H \varpi^\pm = \nabla_{\mathbb{R}^{2n}} \varpi^\pm = \pm \frac{\partial}{\partial \rho} \left(2 \frac{\sqrt{1 - \rho^2}}{\rho} \right) \frac{z}{\rho} = \mp \left(\frac{2}{\rho^2 \sqrt{1 - \rho^2}} \right) \frac{z}{\rho}$$

and hence

$$\frac{\partial \varpi}{\partial \nu_H^\perp} = \mp \frac{2}{\rho^2 \sqrt{1 - \rho^2}} \left\langle \frac{z}{\rho}, (\nu_H^\pm)^\perp \right\rangle = \frac{1}{\rho \sqrt{1 - \rho^2}} \frac{2\sqrt{1 - \rho^2}}{\rho} = \frac{2}{\rho^2}. \quad (9)$$

These computations will be used throughout the next sections in order to study the action of the 2nd order operator \mathcal{L}_{HS} on smooth functions defined on the unit Isoperimetric profile $\mathbb{S}_{\mathbb{H}^n}$.

2.2 Radial case and hypergeometric solutions

In this paper, we will study Problem 1.29 in the case of the model Isoperimetric Profile $\mathbb{S}_{\mathbb{H}^n}$. By applying previous computations, we have

$$\mathcal{L}_{HS} \varphi = \Delta_{HS} \varphi - 2 \frac{\sqrt{1-\rho^2}}{\rho} \left\langle \text{grad}_H \varphi, \left(z^\perp - z \frac{\sqrt{1-\rho^2}}{\rho} \right) \right\rangle;$$

see Section 2.1. In the general case, we have to find \mathbf{C}_{HS}^2 -smooth solutions on $\mathbb{S}_{\mathbb{H}^n} \setminus \{\mathcal{N}, \mathcal{S}\}$ which belong to the space $\Phi(\mathbb{S}_{\mathbb{H}^n})$ of all admissible functions for the horizontal Green's formulas; see Section 1.4. In particular, these functions must belong to the horizontal tangent Sobolev space $\mathcal{H}(S, \sigma_H^{2n})$.

Let $\varphi : \mathbb{S}_{\mathbb{H}^n} \rightarrow \mathbb{R}$ be the restriction of a function $\tilde{\varphi} : \mathbb{H}^n \rightarrow \mathbb{R}$, i.e. $\varphi := \tilde{\varphi}|_{\mathbb{S}_{\mathbb{H}^n}}$, and fix Euclidean cylindrical coordinates on \mathbb{H}^n , i.e. $(\rho, \xi, t) \in \mathbb{R}_+ \times \mathbb{S}^{2n-1} \times \mathbb{R}$, where $\rho = |z|$ and $\xi = \frac{z}{|z|}$. Hereafter, the exponential coordinates $(z, t) \in \mathbb{R}^{2n+1}$ of each point $p \in \mathbb{H}^n$ will be identified with the triple (ρ, ξ, t) .

Remark 2.1. Since $\mathbb{S}_{\mathbb{H}^n}$ is the union of two T -graphs, if $p \in \mathbb{S}_{\mathbb{H}^n}$ then either $p \in \mathbb{S}_{\mathbb{H}^n}^+$ or $p \in \mathbb{S}_{\mathbb{H}^n}^-$. If $p = (z, t) \in \mathbb{S}_{\mathbb{H}^n}^\pm$, one must have $t = \pm u_0(\rho)$; see (8). Using spherical coordinates $(\rho, \xi) \in \mathbb{R}_+ \times \mathbb{S}^{2n-1}$ on \mathbb{R}^{2n} , we get that any real valued function defined on $\mathbb{S}_{\mathbb{H}^n}^\pm$ can be seen as a function of the variables $(\rho, \xi) \in [0, 1] \times \mathbb{S}^{2n-1}$. In the sequel, for every $\varphi : \mathbb{S}_{\mathbb{H}^n} \rightarrow \mathbb{R}$ we shall set $\varphi^\pm := \varphi|_{\mathbb{S}_{\mathbb{H}^n}^\pm}$ and assume that the restrictions $\varphi^\pm : [0, 1] \times \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$ satisfy the “compatibility”, or continuity, constraint: $\varphi^+(1, \xi) = \varphi^-(1, \xi)$ for every $\xi \in \mathbb{S}^{2n-1}$.

Because the radial symmetry of $S = \mathbb{S}_{\mathbb{H}^n}$, in this section we shall preliminarily study the case of radial functions. Below, radial derivatives will be denoted by $\varphi' := \frac{\partial \varphi}{\partial \rho}$ and $\varphi'' := \frac{\partial^2 \varphi}{\partial \rho^2}$.

Notation 2.2. We set $g_H := \langle z, \nu_H \rangle$ and $g_H^\perp := \langle z, \nu_H^\perp \rangle$. The function g_H is called horizontal support function associated with $\mathbb{S}_{\mathbb{H}^n}$.

Under the previous assumptions, we first compute

$$\begin{aligned} \Delta_{HS} \varphi &= \sum_{i \in I_{HS}} \tau_i(\tau_i(\varphi)) = \sum_{i \in I_{HS}} \tau_i \langle \text{grad}_H \varphi, \tau_i \rangle = \sum_{i \in I_{HS}} \tau_i \left\langle \varphi' \frac{z}{\rho}, \tau_i \right\rangle \\ &= \sum_{i \in I_{HS}} \left(\left\langle \nabla_{\tau_i}^H \left(\varphi' \frac{z}{\rho} \right), \tau_i \right\rangle + \left\langle \varphi' \frac{z}{\rho}, \nabla_{\tau_i}^H \tau_i \right\rangle \right) \\ &= \mathcal{H}_H \left(\frac{\varphi'}{\rho} \right) g_H + \sum_{i \in I_{HS}} \tau_i \left(\frac{\varphi'}{\rho} \right) \langle z, \tau_i \rangle + \left(\frac{\varphi'}{\rho} \right) \sum_{i \in I_{HS}} \langle \nabla_{\tau_i}^H z, \tau_i \rangle \\ &= \mathcal{H}_H \left(\frac{\varphi'}{\rho} \right) g_H + \sum_{i \in I_{HS}} \left\langle \text{grad}_H \left(\frac{\varphi'}{\rho} \right), \tau_i \right\rangle \langle z, \tau_i \rangle + \underbrace{(2n-1)}_{=\dim HS} \left(\frac{\varphi'}{\rho} \right) \\ &= \left(\frac{\varphi'}{\rho} \right) (\mathcal{H}_H g_H + 2n - 1) + \frac{\varphi'' \rho - \varphi'}{\rho^2} \sum_{i \in I_{HS}} \left\langle \frac{z}{\rho}, \tau_i \right\rangle \langle z, \tau_i \rangle \\ &= \left(\frac{\varphi'}{\rho} \right) (\mathcal{H}_H g_H + 2n - 1) + \frac{\varphi'' \rho - \varphi'}{\rho^3} \sum_{i \in I_{HS}} \langle z, \tau_i \rangle^2 \\ &= \left(\frac{\varphi'}{\rho} \right) (\mathcal{H}_H g_H + 2n - 1) + \frac{\varphi'' \rho - \varphi'}{\rho^3} (\rho^2 - g_H^2). \end{aligned}$$

Using the last computation together with the identity $\text{grad}_H \varphi = \varphi' \frac{z}{\rho}$ yields

$$\begin{aligned} \mathcal{L}_{HS} \varphi &= \left(\frac{\varphi'}{\rho} \right) (\mathcal{H}_H g_H + 2n - 1) + \frac{\varphi'' \rho - \varphi'}{\rho^3} (\rho^2 - g_H^2) + 2\varphi' \frac{1 - \rho^2}{\rho} \\ &= \frac{(\varphi'' \rho - \varphi')}{\rho} \left(1 - \left(\frac{g_H}{\rho} \right)^2 \right) + \frac{\varphi'}{\rho} (\mathcal{H}_H g_H + 2n + 1 - 2\rho^2). \end{aligned}$$

Since $g_H = \rho^2$ and $\mathcal{H}_H = -2n$, we get that

$$\mathcal{L}_{HS} \varphi = \left(\varphi'' - \frac{\varphi'}{\rho} \right) (1 - \rho^2) + \frac{\varphi'}{\rho} ((Q - 1) - Q\rho^2) = \varphi''(1 - \rho^2) + \frac{\varphi'}{\rho} (2n - (2n + 1)\rho^2).$$

The last one is an interesting ordinary 2nd order differential operator of hypergeometric type. Hence, under our current assumptions, Problem 1.29 reduces to study a 1-dimensional eigenvalue problem

$$\varphi''(1 - \rho^2) + \frac{\varphi'}{\rho} (2n - (2n + 1)\rho^2) = -\lambda\varphi \quad (\lambda \in \mathbb{R}_+) \quad (10)$$

subject to the integral condition $\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi \sigma_H^{2n} = 0$.

Remark 2.3. Instead of φ , we will study its restrictions φ^\pm to the North and South hemispheres $\mathbb{S}_{\mathbb{H}^n}^\pm$ and suppose that φ^\pm are smooth solutions to (10) on $]0, 1[\subset \mathbb{R}$, i.e. $\varphi^\pm \in \mathbf{C}^2([0, 1])$. Moreover, φ^\pm must satisfy the continuity constraint, i.e. $\varphi^+(1, \xi) = \varphi^-(1, \xi)$ for every $\xi \in \mathbb{S}^{2n-1}$, together with the integral condition $\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi \sigma_H^{2n} = 0$. The last one can be translated into an equation involving 1-dimensional weighted integrals of φ^\pm . More precisely, by using the symmetry of $\mathbb{S}_{\mathbb{H}^n}$ with respect to the horizontal hyperplane $t = 0$, we get that

$$\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi \sigma_H^{2n} = \int_{\mathbb{S}_{\mathbb{H}^n}^+} \varphi^+ \sigma_H^{2n} + \int_{\mathbb{S}_{\mathbb{H}^n}^-} \varphi^- \sigma_H^{2n}.$$

Since

$$\sigma_H^{2n} \llcorner \mathbb{S}_{\mathbb{H}^n}^\pm = \frac{\rho}{2\sqrt{1-\rho^2}} dz \llcorner B_1(0),$$

we obtain

$$\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi \sigma_H^{2n} = \int_{\mathbb{S}_{\mathbb{H}^n}^+} \varphi^+ \sigma_H^{2n} + \int_{\mathbb{S}_{\mathbb{H}^n}^-} \varphi^- \sigma_H^{2n} = \frac{O_{2n-1}}{2} \int_0^1 (\varphi^+ + \varphi^-)(\rho) \frac{\rho^{2n}}{\sqrt{1-\rho^2}} d\rho = 0.$$

It follows that any radial function $\varphi : \mathbb{S}_{\mathbb{H}^n} \rightarrow \mathbb{R}$ belongs to $\mathcal{H}(S, \sigma_H^{2n})$ if, and only if, its restrictions φ^\pm satisfy:

$$\varphi^\pm \in L^2 \left([0, 1], \frac{\rho^{2n}}{2\sqrt{1-\rho^2}} d\rho \right), \quad (\varphi^\pm)' \in L^2 \left([0, 1], \frac{\rho^{2n}}{2\sqrt{1-\rho^2}} d\rho \right). \quad (11)$$

Hence, in the radial case, we can reformulate our closed eigenvalue problem as follows:

Problem 2.4 (Radial version). Find all positive real numbers $\lambda \in \mathbb{R}_+$ such that there exist functions $\varphi^+, \varphi^- \in \mathbf{C}^2([0, 1])$ satisfying (11), which are non trivial solutions to:

$$\begin{cases} \varphi''(1 - \rho^2) + \frac{\varphi'}{\rho} (2n - (2n + 1)\rho^2) = -\lambda\varphi & (\lambda \in \mathbb{R}_+) \\ \varphi^+(1) = \varphi^-(1) \\ \int_0^1 (\varphi^+ + \varphi^-)(\rho) \frac{\rho^{2n}}{\sqrt{1-\rho^2}} d\rho = 0. \end{cases}$$

This one can be regarded as a Sturm-Liouville problem with mixed conditions for a hypergeometric O.D.E. The first step is finding the general integral of equation (10).

Notation 2.5. Later on we will set

$$\mathbf{F}(a, b, c, z) := \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k \quad (z \in \mathbb{C}, |z| < 1)$$

where $(d)_k := \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\dots(d+k-1)$ and Γ denotes the Euler Gamma function. The series $\mathbf{F}(a, b, c, z)$ is the so-called hypergeometric series; see [44].

Lemma 2.6. *The general integral $\varphi(\rho) := \varphi(c_1, c_2, \lambda, \rho)$ of (10) is given by*

$$\begin{aligned}\varphi(\rho) &= c_1 \mathbf{F} \left(\frac{n - \sqrt{n^2 + \lambda}}{2}; \frac{n + \sqrt{n^2 + \lambda}}{2}; \frac{1}{2} + n; \rho^2 \right) \\ &+ c_2 \frac{1}{\rho^{2n-1}} \mathbf{F} \left(\frac{1 - n - \sqrt{n^2 + \lambda}}{2}; \frac{1 - n + \sqrt{n^2 + \lambda}}{2}; \frac{3}{2} - n; \rho^2 \right).\end{aligned}$$

Note that the second function is singular at $\rho = 0$. Further, set

$$\begin{aligned}\psi_1(\lambda, \rho) &:= \mathbf{F} \left(\frac{n - \sqrt{n^2 + \lambda}}{2}; \frac{n + \sqrt{n^2 + \lambda}}{2}; \frac{1}{2} + n; \rho^2 \right), \\ \psi_2(\lambda, \rho) &:= \frac{1}{\rho^{2n-1}} \mathbf{F} \left(\frac{1 - n - \sqrt{n^2 + \lambda}}{2}; \frac{1 - n + \sqrt{n^2 + \lambda}}{2}; \frac{3}{2} - n; \rho^2 \right).\end{aligned}$$

A rigorous (and elementary) proof of the previous lemma uses *Frobenius' method*. Nevertheless, a more concise proof can be done by using the theory of *hypergeometric functions*.

Proof of Lemma 2.6. First, set $s := \rho^2$ on $[0, 1]$ and $\varphi(\rho) := \psi(\rho^2) = \psi(s)$ for $s \in [0, 1]$. Under this transformation, equation (10) becomes

$$s(1-s)\psi'' + \psi' \left(\frac{(Q-1) - Qs}{2} \right) = -\frac{\lambda}{4}\psi. \quad (12)$$

Indeed, from the elementary identities $\varphi'(\rho) = 2\rho\psi'(\rho^2)$, $\varphi''(\rho) = 4\rho^2\psi''(\rho^2) + 2\psi'(\rho^2)$, we get

$$4s(1-s)\psi'' + ((2n+2)(1-s) - s)\psi' = -\lambda\psi$$

and it is enough to divide by 4 and to use $Q = 2n + 2$. In this form our equation can be handled by means of the general theory of hypergeometric functions, for which we refer the reader to the book by Wang and Guo [44]. In particular, by using the notation of Chapter 4 in [44], the general hypergeometric equation (of the complex variable z) is given by

$$z(1-z)\psi'' + (\gamma - (\alpha + \beta + 1))\psi' - \alpha\beta\psi = 0. \quad (13)$$

This equation has two linearly independent solutions, which can be written out in terms of hypergeometric series (see Notation 2.5), i.e.

$$\psi_1(z) := \mathbf{F}(\alpha; \beta; \gamma; z), \quad \psi_2(z) := \frac{1}{z^{\gamma-1}} \mathbf{F}(\alpha - \gamma + 1; \beta - \gamma + 1; 2 - \gamma; z).$$

By analyzing the coefficients of equation (12), we find that $\gamma = \frac{Q-1}{2}$ and so

$$\alpha + \beta + 1 = \frac{Q}{2} \quad \alpha\beta = -\frac{\lambda}{4}.$$

Hence

$$\alpha = \frac{n - \sqrt{n^2 + \lambda}}{2}, \quad \beta = \frac{n + \sqrt{n^2 + \lambda}}{2}, \quad \gamma = n + \frac{1}{2},$$

and the thesis easily follows. \square

We are now in a position to solve Problem 2.4. Remind that we are looking for solutions $\varphi^\pm \in \mathbf{C}^2([0, 1])$ satisfying (11). So let us consider the second solution ψ_2 . It is elementary to see that $\psi_2(\lambda, \rho) \sim \frac{1}{\rho^{2n-1}}$ as long as $\rho \rightarrow 0^+$ because the hypergeometric series, evaluated at $\rho = 0$, takes the value 1. Therefore ψ_2 cannot satisfy the conditions in (11) and we get

$$\varphi^\pm(\rho) = c^\pm \psi_1(\lambda, \rho) \quad \rho \in [0, 1],$$

for some real constant c^\pm . We have now to use the “mixed” conditions of Problem 2.4. We see that the third condition, together with the continuity of φ , implies the following two possibilities:

- (i) $\varphi^+(\rho) = \varphi^-(\rho)$ for every $\rho \in [0, 1]$;
- (ii) $\varphi^+(\rho) = -\varphi^-(\rho)$ for every $\rho \in [0, 1]$ and $\varphi^+(1) = \varphi^-(1) = 0$.

Lemma 2.7. *In case (i) holds, Problem 2.4 admits a countable family of eigenvalues which are given by $\lambda_{2m} := 2m(2m + 2n)$, $m \in \mathbb{N}$. The eigenfunction φ_{2m} relative to λ_{2m} is polynomial and explicitly given, for some real constant c , by*

$$\varphi_{2m}(\rho) := c \mathbf{F} \left(-m; n + m; \frac{1}{2} + n; \rho^2 \right).$$

Proof. Since $\varphi^+ = \varphi^-$ on $[0, 1]$, it is enough to write down the integral condition. Hence we have to find all positive real numbers $\lambda \in \mathbb{R}_+$ such that any non-singular solution φ to (10) (i.e. $\varphi = c\psi_1(\lambda, \rho)$ for some $c \in \mathbb{R}$) satisfies

$$\int_0^1 \varphi(\rho) \frac{\rho^{2n}}{\sqrt{1 - \rho^2}} d\rho = 0.$$

By using some classical results about hypergeometric series (or MATHEMATICA) we see that

$$\int_0^1 \psi_1(\lambda, \rho) \frac{\rho^{2n}}{\sqrt{1 - \rho^2}} d\rho = \frac{\sqrt{\pi} \Gamma(n + \frac{1}{2})}{2 \Gamma(\frac{2+n-\sqrt{n^2+\lambda}}{2}) \Gamma(\frac{2+n+\sqrt{n^2+\lambda}}{2})}.$$

It remains to solve (in the variable $\lambda \geq 0$) the following equation:

$$\frac{\sqrt{\pi} \Gamma(n + \frac{1}{2})}{2 \Gamma(\frac{2+n-\sqrt{n^2+\lambda}}{2}) \Gamma(\frac{2+n+\sqrt{n^2+\lambda}}{2})} = 0. \quad (14)$$

From standard properties of the Gamma function we easily get that $\frac{2+n-\sqrt{n^2+\lambda}}{2}$ must be a negative integer or 0, i.e.

$$\frac{2 + n - \sqrt{n^2 + \lambda}}{2} = -m, \quad m \in \mathbb{N}.$$

Therefore $\lambda = 2(m + 1)(2(m + 1) + 2n)$ for any $m \in \mathbb{N}$. From now on, we shall set

$$\lambda_{2m} := 2m(2m + 2n), \quad m \in \mathbb{N}_+.$$

Remark 2.8. *In this list, we have not included the eigenvalue $\lambda_0 = 0$ because, as already said, the only eigenfunction relative to λ_0 is $\varphi_0 = 0$.*

The eigenfunction φ_{2m} relative to the eigenvalue λ_{2m} ($m \geq 1$) is given, up to a real constant c , by

$$\varphi_{2m}(\rho) := c\psi_1(\lambda_{2m}, \rho) = c \mathbf{F} \left(-m; n + m; \frac{1}{2} + n; \rho^2 \right).$$

From the general theory of hypergeometric series it follows that φ_{2m} is a polynomial function. By construction, $\varphi_{2m} = \varphi_{2m}^+ = \varphi_{2m}^-$ and we have

$$\int_0^1 \varphi_{2m}(\rho) \frac{\rho^{2n}}{\sqrt{1 - \rho^2}} d\rho = 0.$$

□

Lemma 2.9. *In case (ii) holds, Problem 2.4 admits a countable family of eigenvalues which are given by $\lambda_{2m+1} := (2m + 1)((2m + 1) + 2n)$, $m \in \mathbb{N}$. The eigenfunction φ_{2m+1} relative to λ_{2m+1} is the hypergeometric function given by*

$$\varphi_{2m+1}(\rho) := c \mathbf{F} \left(-m - \frac{1}{2}; n + m + \frac{1}{2}; \frac{1}{2} + n; \rho^2 \right),$$

for some real constant c . Furthermore, we have $\varphi_{2m+1} = \varphi_{2m+1}^+$ on $\mathbb{S}_{\mathbb{H}^n}^+$, $\varphi_{2m+1} = \varphi_{2m+1}^-$ on $\mathbb{S}_{\mathbb{H}^n}^-$ and $\varphi_{2m+1}^+ = -\varphi_{2m+1}^-$.

Proof. We are assuming that $\varphi^+ = -\varphi^-$ on $[0, 1]$ and that $\varphi^+(1) = -\varphi^-(1) = 0$. So we have to find all positive real numbers $\lambda \in \mathbb{R}_+$ such that any pair of non-singular solutions φ^\pm to (10) (i.e. $\varphi^\pm = c^\pm \psi_1(\lambda, \rho)$, for some $c^\pm \in \mathbb{R}$) satisfies:

$$\varphi^+(1) = -\varphi^-(1) = 0.$$

In particular, we get that $c^+ = -c^-$ and it remains to solve (in the variable $\lambda \geq 0$) the equation $\psi_1(\lambda, \rho) = 0$, i.e.

$$\mathbf{F}\left(\frac{n - \sqrt{n^2 + \lambda}}{2}; \frac{n + \sqrt{n^2 + \lambda}}{2}; \frac{1}{2} + n; 1\right) = 0.$$

We recall that a classical result about hypergeometric series states that

$$\mathbf{F}(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}$$

as long as $\Re(\gamma - \alpha - \beta) > 0$; see [44] Chapter 4.7, p. 156. Note that $\gamma - \alpha - \beta = \frac{1}{2} > 0$. So we have to find the values λ such that

$$\frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} = 0. \quad (15)$$

The general theory of the Gamma function (see [44], Chapter 3) says that this equation can be solved if, and only if, $\gamma - \alpha$ or $\gamma - \beta$ are negative integers, or 0. This in turn implies

$$\gamma - \beta = \frac{n + 1 - \sqrt{n^2 + \lambda}}{2} = -m, \quad m \in \mathbb{N}.$$

So we get $\lambda_{2m+1} := (2m + 1)(2m + 1 + 2n)$, $m \in \mathbb{N}$. The eigenfunction φ_{2m+1} relative to the eigenvalue λ_{2m+1} ($m \in \mathbb{N}$) is given, up to a real constant c , by

$$\varphi_{2m+1}(\rho) := c \psi_1(\lambda_{2m+1}, \rho) = c \mathbf{F}\left(-m - \frac{1}{2}; n + m + \frac{1}{2}; \frac{1}{2} + n; \rho^2\right).$$

The last one is a hypergeometric function and, by construction, $\varphi_{2m+1}^+ = -\varphi_{2m+1}^-$. Therefore

$$\int_0^1 (\varphi_{2m+1}^+ + \varphi_{2m+1}^-)(\rho) \frac{\rho^{2n}}{\sqrt{1 - \rho^2}} d\rho = 0. \quad (16)$$

□

Theorem 2.10. *Problem 2.4 admits a countable family of positive real eigenvalues*

$$\lambda_k := k(k + 2n), \quad k \in \mathbb{N}.$$

Each eigenvalue λ_k has multiplicity 1 and its associated eigenfunction φ_k , up to real constants, is a hypergeometric function. More precisely, we have the following:

(i) *if $k = 2m$, the eigenfunction φ_{2m} associated with λ_{2m} is the hypergeometric polynomial given by*

$$\varphi_{2m}(\rho) := c \mathbf{F}\left(-m; n + m; \frac{1}{2} + n; \rho^2\right),$$

for some real constant c . Moreover, $\varphi_{2m} = \varphi_{2m}^+ = \varphi_{2m}^-$ and

$$\int_0^1 \varphi_{2m}(\rho) \frac{\rho^{2n}}{\sqrt{1 - \rho^2}} d\rho = 0.$$

(ii) *if $k = 2m + 1$, the eigenfunction φ_{2m+1} associated with λ_{2m+1} is the hypergeometric function*

$$\varphi_{2m+1}(\rho) := c \mathbf{F}\left(-m - \frac{1}{2}; n + m + \frac{1}{2}; \frac{1}{2} + n; \rho^2\right),$$

for some real constant c . Moreover, $\varphi_{2m+1}^+ = -\varphi_{2m+1}^-$ and (16) holds true.

Proof. Applying Lemma 2.7 and Lemma 2.9. \square

Corollary 2.11. *Up to real constants, the eigenfunction φ_1 associated with the first eigenvalue $\lambda_1 = Q - 1$ of Problem 2.4 is given by*

$$\varphi_1(\rho) = \sqrt{1 - \rho^2} \quad \text{for every } \rho \in [0, 1].$$

Proof. This follows from (ii), by using the identity

$$(1 + z)^\alpha = \mathbf{F}(-\alpha, \beta, \beta, -z),$$

and by setting $\alpha = \frac{1}{2}$ and $z = -\rho^2$; see, for instance, [44], p. 137, formula (10). \square

Remark 2.12. *Let $S \subset \mathbb{H}^n$ be a smooth compact hypersurface without boundary. We stress that*

$$\varphi_2(p) := (2n - 1) + \mathcal{H}_H g_H - \varpi g_H^\perp,$$

is a zero-mean function, i.e. $\int_S \varphi_2 \sigma_H^{2n} = 0$, where $g_H = \langle z, \nu_H \rangle$ and $g_H^\perp = \langle z, \nu_H^\perp \rangle$. Furthermore if $S = \mathbb{S}_{\mathbb{H}^n}$, then $\varphi_2(\rho) = (Q - 1) - Q\rho^2$ turns out to be radial. We also stress that, in this case, the function φ_2 is, up to real constants, the 2nd radial eigenfunction of Problem 2.4, corresponding to $\lambda_2 = 2Q$.

2.3 Some remarks about the general case

Warning 2.13. *From now on, when we refer to Problem 1.29 it is understood that $S = \mathbb{S}_{\mathbb{H}^n}$.*

Remark 2.14. *As already said, any function $\varphi : \mathbb{S}_{\mathbb{H}^n} \setminus \{\mathcal{N}, \mathcal{S}\} \rightarrow \mathbb{R}$ can be regarded as a restriction of some function $\tilde{\varphi} : \mathbb{H}^n \rightarrow \mathbb{R}$. Moreover, $\mathbb{S}_{\mathbb{H}^n} = \mathbb{S}_{\mathbb{H}^n}^+ \cup \mathbb{S}_{\mathbb{H}^n}^-$ and the hemispheres $\mathbb{S}_{\mathbb{H}^n}^\pm$ are the radial T -graphs given by $t = \pm u_0(\rho)$, where $\rho = |z|$, $z \in \overline{B_1(0)}$, and u_0 was defined by formula (8). By introducing polar coordinates¹² $(\rho, \xi, t) \in \mathbb{R}_+ \cup \{0\} \times \mathbb{S}^{2n-1} \times \mathbb{R}$ on \mathbb{H}^n and by setting*

$$\Phi^\pm : \overline{B_1(0)} \rightarrow \mathbb{H}^n, \quad \Phi^\pm(\rho, \xi) := (\rho, \xi, \pm u_0(\rho)),$$

it follows that $\varphi^\pm := \tilde{\varphi} \circ \Phi^\pm$ only depends on the variables (ρ, ξ) .

Remember also that

$$\Delta_{HS} \varphi = \Delta_H \varphi + \mathcal{H}_H \frac{\partial \varphi}{\partial \nu_H} - \langle \text{Hess}_H \varphi \nu_H, \nu_H \rangle;$$

see Lemma 1.15. So if $\varphi^\pm = \tilde{\varphi} \circ \Phi^\pm : \overline{B_1(0)} \setminus \{0\} \rightarrow \mathbb{R}$, by a simple computation we get that:

- $\Delta_H \varphi^\pm = \Delta_{\mathbb{R}^{2n}} \varphi^\pm$,
- $\frac{\partial^\pm \varphi}{\partial \nu_H^\pm} = \pm \langle \nabla_{\mathbb{R}^{2n}} \varphi^\pm, (z \pm \kappa z^\perp) \rangle$,
- $\langle \text{Hess}_H \varphi^\pm \nu_H, \nu_H \rangle = \langle \text{Hess}_{\mathbb{R}^{2n}} \varphi^\pm (z \pm \kappa z^\perp), (z \pm \kappa z^\perp) \rangle$,

where $\nu_H^\pm = z \pm \kappa z^\perp$ and we have set $\kappa(\rho) := \frac{\sqrt{1-\rho^2}}{\rho}$. Because the symmetry of $\mathbb{S}_{\mathbb{H}^n}$ with respect to the horizontal hyperplane $H_0 \cong \mathbb{R}^{2n}$, we will carry out the computations only for the north hemisphere $\mathbb{S}_{\mathbb{H}^n}^+ \setminus \{\mathcal{N}\}$. For brevity, set $\varphi = \varphi^+$ and $\nu_H = \nu_H^+$. We have

$$\Delta_{HS} \varphi = \Delta_{\mathbb{R}^{2n}} \varphi - 2n \left(\frac{\partial \varphi}{\partial z} + \kappa \frac{\partial \varphi}{\partial z^\perp} \right) - \langle \text{Hess}_H \varphi \nu_H, \nu_H \rangle,$$

where we have used $\mathcal{H}_H|_{\mathbb{S}_{\mathbb{H}^n}} = -2n$. Since

$$\mathcal{L}_{HS} \varphi = \Delta_{HS} \varphi + 2\kappa \left(\kappa \frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial z^\perp} \right),$$

we obtain

$$\mathcal{L}_{HS} \varphi = \Delta_{\mathbb{R}^{2n}} \varphi - 2(n - \kappa^2) \frac{\partial \varphi}{\partial z} - Q\kappa \frac{\partial \varphi}{\partial z^\perp} - \langle \text{Hess}_H \varphi \nu_H, \nu_H \rangle. \quad (17)$$

¹²This means that we are fixing “ordinary” spherical coordinates $(\rho, \xi) \in \mathbb{R}_+ \cup \{0\} \times \mathbb{S}^{2n-1}$ on $\mathbb{R}^{2n} \cong H_0$ and so $z \in \mathbb{R}^{2n}$ corresponds to $(\rho, \xi) \in \mathbb{R}_+ \cup \{0\} \times \mathbb{S}^{2n-1}$.

Notation 2.15. Set

$$\zeta := \frac{z^\perp}{\rho} \in T\mathbb{S}^{2n-1}.$$

Hereafter, either $\frac{\partial \varphi}{\partial \rho}$ or φ'_ρ , will denote the 1st partial derivative of φ with respect to the variable $\rho = |z|$. Moreover, either $\frac{\partial \varphi}{\partial \zeta}$ or φ'_ζ will denote the 1st partial derivative of φ with respect to the “angular” variable ζ . An analogous notation will be used for 2nd partial derivatives.

Lemma 2.16. We have $\operatorname{div}_{T\mathbb{S}^{2n-1}} \zeta = 0$.

Proof. We claim that

$$\operatorname{div}_{T\mathbb{S}^{2n-1}} \zeta = \operatorname{div}_{\mathbb{R}^{2n}} \zeta - \sum_{i=1}^{2n} \left\langle \operatorname{grad}_{\mathbb{R}^{2n}} \zeta_i, \frac{z}{\rho} \right\rangle \left\langle \frac{z}{\rho}, e_i \right\rangle,$$

where $\zeta_i = \langle \zeta, e_i \rangle$ for $i = 1, \dots, 2n$. The last formula easily follows by using definitions. Now the first term vanishes because $\operatorname{div}_{\mathbb{R}^{2n}} z^\perp = 0$. Using

$$\mathcal{J}_{\mathbb{R}^{2n}} \zeta = -\frac{1}{\rho} C_H^{2n+1} - \frac{1}{\rho^4} \zeta \otimes z,$$

yields

$$\sum_{i=1}^{2n} \left\langle \operatorname{grad}_{\mathbb{R}^{2n}} \zeta_i, \frac{z}{\rho} \right\rangle \left\langle \frac{z}{\rho}, e_i \right\rangle = \left\langle \mathcal{J}_{\mathbb{R}^{2n}} \zeta, \frac{z}{\rho} \right\rangle = 0.$$

□

Lemma 2.17. Under the previous assumptions, one has

$$\langle \operatorname{Hess}_H \varphi \nu_H, \nu_H \rangle = \rho^2 \varphi''_{\rho\rho} + (1 - \rho^2) \varphi''_{\rho\zeta} + \rho \sqrt{1 - \rho^2} \varphi''_{\zeta\zeta} + \frac{1 - \rho^2}{\rho} \varphi'_\rho - \sqrt{1 - \rho^2} \varphi'_\zeta.$$

Proof. By using Lemma 1.17 we have

$$\langle \operatorname{Hess}_H \varphi \nu_H, \nu_H \rangle = \frac{\partial^2 \varphi}{\partial \nu_H^2} + \left\langle \frac{\operatorname{grad}_{HS} \varpi}{\varpi}, \operatorname{grad}_{HS} \varphi \right\rangle - \varpi \frac{\partial \varphi}{\partial \nu_H^\perp}. \quad (18)$$

We compute

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial \nu_H^2} &= \frac{\partial}{\partial \nu_H} \left(\frac{\partial \varphi}{\partial z} + \kappa \frac{\partial \varphi}{\partial z^\perp} \right) \\ &= \frac{\partial}{\partial \nu_H} \left(\rho \frac{\partial \varphi}{\partial \rho} + \sqrt{1 - \rho^2} \frac{\partial \varphi}{\partial \zeta} \right) \\ &= \frac{\partial}{\partial z} \left(\rho \frac{\partial \varphi}{\partial \rho} + \sqrt{1 - \rho^2} \frac{\partial \varphi}{\partial \zeta} \right) + \kappa \frac{\partial}{\partial z^\perp} \left(\rho \frac{\partial \varphi}{\partial \rho} + \sqrt{1 - \rho^2} \frac{\partial \varphi}{\partial \zeta} \right) \\ &= \rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \varphi}{\partial \rho} + \sqrt{1 - \rho^2} \frac{\partial \varphi}{\partial \zeta} \right) + \sqrt{1 - \rho^2} \frac{\partial}{\partial \zeta} \left(\rho \frac{\partial \varphi}{\partial \rho} + \sqrt{1 - \rho^2} \frac{\partial \varphi}{\partial \zeta} \right). \end{aligned}$$

From this we easily get that

$$\frac{\partial^2 \varphi}{\partial \nu_H^2} = \rho^2 \varphi''_{\rho\rho} + \rho \varphi'_\rho - \frac{\rho^2}{\sqrt{1 - \rho^2}} \varphi'_\zeta + 2\rho \sqrt{1 - \rho^2} \varphi''_{\zeta\rho} + (1 - \rho^2) \varphi''_{\zeta\zeta}. \quad (19)$$

On the other hand, using some computations made in Section 2.1, yields

$$\frac{\operatorname{grad}_{HS} \varpi}{\varpi} = -\frac{z_{HS}}{\rho^2(1 - \rho^2)}.$$

Moreover, one has

$$z_{HS} = z - \langle z, \nu_H \rangle \nu_H = z - \rho^2 \nu_H = z(1 - \rho^2) - \rho \sqrt{1 - \rho^2} z^\perp.$$

Therefore

$$\begin{aligned} & \left\langle \frac{\text{grad}_{HS} \varpi}{\varpi}, \text{grad}_{HS} \varphi \right\rangle - \varpi \frac{\partial \varphi}{\partial \nu_H^\perp} \\ &= - \left\langle \left(\frac{z(1 - \rho^2) - \rho \sqrt{1 - \rho^2} z^\perp}{\rho^2(1 - \rho^2)} + 2 \frac{\sqrt{1 - \rho^2}}{\rho} (z^\perp - \kappa z) \right), \text{grad}_H \varphi \right\rangle \\ &= -(2\rho^2 - 1) \left\langle \left(\frac{z}{\rho^2} - \frac{z^\perp}{\rho \sqrt{1 - \rho^2}} \right), \text{grad}_H \varphi \right\rangle \\ &= -(2\rho^2 - 1) \left(\frac{1}{\rho} \frac{\partial \varphi}{\partial \rho} - \frac{1}{\sqrt{1 - \rho^2}} \frac{\partial \varphi}{\partial \zeta} \right) \\ &= \frac{1 - 2\rho^2}{\rho} \varphi'_\rho - \frac{1 - 2\rho^2}{\sqrt{1 - \rho^2}} \varphi'_\zeta. \end{aligned}$$

Using (18), (19) and the last computation yields

$$\begin{aligned} & \langle \text{Hess}_H \varphi \nu_H, \nu_H \rangle \\ &= \rho^2 \varphi''_{\rho\rho} + \rho \varphi'_\rho - \frac{\rho^2}{\sqrt{1 - \rho^2}} \varphi'_\zeta + 2\rho \sqrt{1 - \rho^2} \varphi''_{\zeta\rho} + (1 - \rho^2) \varphi''_{\zeta\zeta} + \frac{1 - 2\rho^2}{\rho} \varphi'_\rho - \frac{1 - 2\rho^2}{\sqrt{1 - \rho^2}} \varphi'_\zeta \\ &= \rho^2 \varphi''_{\rho\rho} + (1 - \rho^2) \varphi''_{\zeta\zeta} + 2\rho \sqrt{1 - \rho^2} \varphi''_{\zeta\rho} + \frac{1 - \rho^2}{\rho} \varphi'_\rho - \sqrt{1 - \rho^2} \varphi'_\zeta \end{aligned}$$

which achieves the proof. \square

Remind that the Euclidean Laplace operator in spherical coordinates is given by

$$\Delta_{\mathbb{R}^{2n}} \varphi = \frac{1}{\rho^{2n-1}} \frac{\partial}{\partial \rho} \left(\rho^{2n-1} \frac{\partial \varphi}{\partial \rho} \right) + \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} (\varphi|_{\mathbb{S}^{2n-1}(\rho)}),$$

where $\Delta_{\mathbb{S}^{2n-1}}$ denotes the Laplacian on the Sphere \mathbb{S}^{2n-1} . For the sake of simplicity, we will set

$$\Delta_{\mathbb{S}^{2n-1}} \varphi = \Delta_{\mathbb{S}^{2n-1}} (\varphi|_{\mathbb{S}^{2n-1}(\rho)}).$$

In particular, we have

$$\Delta_{\mathbb{R}^{2n}} \varphi = \varphi''_{\rho\rho} + \frac{2n-1}{\rho} \varphi'_\rho + \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \varphi.$$

From (17), Lemma 2.17 and the last formula we finally obtain

$$\begin{aligned} \mathcal{L}_{HS} \varphi &= \Delta_{\mathbb{R}^{2n}} \varphi - 2\rho(n - \kappa^2) \varphi'_\rho - Q\rho\kappa \varphi'_\zeta - \langle \text{Hess}_H \varphi \nu_H, \nu_H \rangle \\ &= \varphi''_{\rho\rho} + \frac{2n-1}{\rho} \varphi'_\rho + \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \varphi - 2\rho(n - \kappa^2) \varphi'_\rho - Q\sqrt{1 - \rho^2} \varphi'_\zeta \\ &\quad - \left(\rho^2 \varphi''_{\rho\rho} + (1 - \rho^2) \varphi''_{\zeta\zeta} + 2\rho \sqrt{1 - \rho^2} \varphi''_{\zeta\rho} + \frac{1 - \rho^2}{\rho} \varphi'_\rho - \sqrt{1 - \rho^2} \varphi'_\zeta \right) \\ &= (1 - \rho^2) \varphi''_{\rho\rho} + \frac{2n - (2n+1)\rho^2}{\rho} \varphi'_\rho - 2\rho \sqrt{1 - \rho^2} \varphi''_{\zeta\rho} \\ &\quad + \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \varphi - (1 - \rho^2) \varphi''_{\zeta\zeta} - (Q - 1) \sqrt{1 - \rho^2} \varphi'_\zeta. \end{aligned}$$

We resume the previous computations in the next:

Proposition 2.18. Let $\varphi : \overline{B_1(0)} \setminus \{0\} \rightarrow \mathbb{R}$ be a \mathbf{C}^2 -smooth function; then the operator \mathcal{L}_{HS} takes the following form:

$$\begin{aligned} \mathcal{L}_{HS} \varphi &= \underbrace{(1 - \rho^2)\varphi''_{\rho\rho} + \frac{2n - (2n+1)\rho^2}{\rho}\varphi'_\rho}_{\text{Radial Operator}} - \underbrace{2\rho\sqrt{1 - \rho^2}\varphi''_{\zeta\rho}}_{\text{Mixed Derivatives}} \\ &\quad + \underbrace{\frac{1}{\rho^2}\Delta_{\mathbb{S}^{2n-1}}\varphi - (1 - \rho^2)\varphi''_{\zeta\zeta} - (Q - 1)\sqrt{1 - \rho^2}\varphi'_\zeta}_{\text{Angular Operator}}. \end{aligned} \quad (20)$$

Equivalently, one has

$$\begin{aligned} \mathcal{L}_{HS} \varphi &= \underbrace{(1 - \rho^2)(\Delta_{\mathbb{R}^{2n}}\varphi - \varphi''_{\zeta\zeta}) - 2\rho\sqrt{1 - \rho^2}\varphi''_{\zeta\rho} + \Delta_{\mathbb{S}^{2n-1}}\varphi}_{\text{2nd order part}} \\ &\quad - \underbrace{2\rho\varphi'_\rho - (Q - 1)\sqrt{1 - \rho^2}\varphi'_\zeta}_{\text{1st order part}}. \end{aligned} \quad (21)$$

Proof. In order to prove formula (21), it is enough using (20) together with the obvious identity

$$(1 - \rho^2)\Delta_{\mathbb{R}^{2n}}\varphi = (1 - \rho^2)\left(\varphi''_{\rho\rho} + \frac{2n-1}{\rho}\varphi'_\rho + \frac{1}{\rho^2}\Delta_{\mathbb{S}^{2n-1}}\varphi\right).$$

□

Remark 2.19 (Case \mathbb{H}^1). If $n = 1$, fix polar coordinates on \mathbb{R}^2 , i.e. $(\rho, \vartheta) \in \mathbb{R}_+ \times [0, 2\pi]$ so that $z = (x, y) = (\rho \cos \vartheta, \rho \sin \vartheta) \in \mathbb{R}^2$. By applying Chain Rule, we get that

$$\frac{\partial \varphi}{\partial \zeta} = \frac{1}{\rho} \frac{\partial \varphi}{\partial \vartheta}, \quad \frac{\partial^2 \varphi}{\partial \zeta^2} = \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \vartheta^2}$$

for every smooth function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$. We also stress that

$$\Delta_{\mathbb{S}^1} \varphi = \varphi''_{\vartheta\vartheta}, \quad \frac{\partial^2 \varphi}{\partial \zeta^2} = \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \vartheta^2}.$$

By using (21), we compute

$$\begin{aligned} \mathcal{L}_{HS} \varphi &= (1 - \rho^2)\Delta_{\mathbb{R}^2}\varphi - 2\rho\sqrt{1 - \rho^2}\varphi''_{\zeta\rho} + \rho^2\varphi''_{\zeta\zeta} - 2\rho\varphi'_\rho - 3\sqrt{1 - \rho^2}\varphi'_\zeta \\ &= (1 - \rho^2)\Delta_{\mathbb{R}^2}\varphi - 2\sqrt{1 - \rho^2}\varphi''_{\vartheta\rho} + \varphi''_{\vartheta\vartheta} - 2\rho\varphi'_\rho - 3\frac{\sqrt{1 - \rho^2}}{\rho}\varphi'_\vartheta. \end{aligned} \quad (22)$$

Equivalently, by expressing the Laplacian $\Delta_{\mathbb{R}^2}$ in polar coordinates, we see that

$$\mathcal{L}_{HS} \varphi = (1 - \rho^2)\varphi''_{\rho\rho} - 2\sqrt{1 - \rho^2}\varphi''_{\vartheta\rho} + \varphi''_{\vartheta\vartheta} + \frac{2 - 3\rho^2}{\rho}\varphi'_\rho - 3\frac{\sqrt{1 - \rho^2}}{\rho}\varphi'_\vartheta. \quad (23)$$

The operator \mathcal{L}_{HS} turns out to be “strictly non-elliptic”. In fact, the determinant of the (2×2) -matrix associated with the 2nd order coefficients is zero at every point of $\overline{B_1(0)} \setminus \{0\}$. In particular, by using (17), we easily see that the 2nd order part of $\mathcal{L}_{HS} \varphi$ is given by

$$\Delta_{\mathbb{R}^2} \varphi - \langle \text{Hess}_{\mathbb{R}^2} \varphi \nu_H, \nu_H \rangle = (\nu_H^2)^2 \frac{\partial^2 \varphi}{\partial x^2} + (\nu_H^1)^2 \frac{\partial^2 \varphi}{\partial y^2} - 2\nu_H^1 \nu_H^2 \frac{\partial^2 \varphi}{\partial x \partial y},$$

where $\nu_H = (\nu_H^1, \nu_H^2)$. Since the matrix

$$\begin{pmatrix} (\nu_H^2)^2 & -\nu_H^1 \nu_H^2 \\ -\nu_H^1 \nu_H^2 & (\nu_H^1)^2 \end{pmatrix},$$

has eigenvalues 0 and 1 at every point of $\overline{B_1(0)} \setminus \{0\}$, it follows that \mathcal{L}_{HS} is of “parabolic-type”.

Our final remarks concern “purely angular” functions. So let $\varphi : \overline{B_1(0)} \setminus \{0\} \rightarrow \mathbb{R}$ be such that $\frac{\partial \varphi}{\partial \rho} = 0$. In case of purely angular functions, Proposition 2.18 says that

$$\mathcal{L}_{HS} \varphi = \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \varphi - (1 - \rho^2) \varphi''_{\zeta\zeta} - (Q - 1) \sqrt{1 - \rho^2} \varphi'_\zeta. \quad (24)$$

Therefore, the eigenvalue equation becomes

$$\frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \varphi - (1 - \rho^2) \varphi''_{\zeta\zeta} - (Q - 1) \sqrt{1 - \rho^2} \varphi'_\zeta = -\lambda \varphi. \quad (25)$$

After multiplying both members of (25) by ρ^2 , we have

$$\Delta_{\mathbb{S}^{2n-1}} \varphi - \rho^2 (1 - \rho^2) \varphi''_{\zeta\zeta} - (Q - 1) \rho^2 \sqrt{1 - \rho^2} \varphi'_\zeta = -\lambda \rho^2 \varphi. \quad (26)$$

Lemma 2.20. *There exists no non-trivial purely angular solution to Problem 1.29.*

Proof. By contradiction. Changing the sign and differentiating (26) with respect to ρ , yields

$$2\rho(1 - 2\rho^2) \varphi''_\zeta + Q \frac{2 - 3\rho^2}{\kappa} \varphi'_\zeta = 2\lambda \rho \varphi.$$

For a fixed $\rho \in]0, 1]$, the last one is a 2nd order O.D.E. whose general solution must depend on ρ . This contradicts the fact that φ was assumed to be purely angular. \square

Remark 2.21. *Let φ_i be the i -th radial eigenfunction of Problem 2.4 associated with λ_i and let μ be an angular function. Note that*

$$\mathcal{L}_{HS}(\varphi_i \mu) = \mathcal{L}_{HS}(\varphi_i) \mu + \varphi_i \mathcal{L}_{HS}(\mu) + 2 \langle \text{grad}_{HS} \varphi_i, \text{grad}_{HS} \mu \rangle.$$

Assume that $\varphi := \varphi_i \mu$ is a solution of Problem 1.29 associated with λ . Then, by applying the previous formula we obtain¹³

$$\lambda_i \varphi_i \mu + \underbrace{\varphi_i \left\{ \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \mu - (1 - \rho^2) \mu''_{\zeta\zeta} - (Q - 1) \sqrt{1 - \rho^2} \mu'_\zeta \right\}}_{=\mathcal{L}_{HS} \mu} - 2\rho \sqrt{1 - \rho^2} (\varphi_i)'_\rho \mu'_\zeta = -\lambda \varphi_i \mu.$$

So let $\rho \in]0, 1[$ be such that $\varphi_i(\rho) \neq 0$. The previous equation gives

$$\mathcal{L}_{HS} \mu - 2 \frac{(\varphi_i)'_\rho \rho \sqrt{1 - \rho^2}}{\varphi_i} \mu'_\zeta = -(\lambda - \lambda_i) \mu.$$

Multiplying by $-\mu$ and integrating the resulting equation along $\mathbb{S}_{\mathbb{H}^n}$ with respect to σ_H^{2n} , yields

$$\int_{\mathbb{S}_{\mathbb{H}^n}} |\text{grad}_{HS} \mu|^2 \sigma_H^{2n} + \int_{\mathbb{S}_{\mathbb{H}^n}} \left(\frac{(\varphi_i)'_\rho \rho \sqrt{1 - \rho^2}}{\varphi_i} \right) \frac{\partial(\mu^2)}{\partial \zeta} \sigma_H^{2n} = (\lambda - \lambda_i) \int_{\mathbb{S}_{\mathbb{H}^n}} \mu^2 \sigma_H^{2n}.$$

Furthermore, note that

$$\int_{\mathbb{S}_{\mathbb{H}^n}} \left(\frac{(\varphi_i)'_\rho \rho \sqrt{1 - \rho^2}}{\varphi_i} \right) \frac{\partial(\mu^2)}{\partial \zeta} \sigma_H^{2n} = 2 \int_0^1 \frac{(\varphi_i)'_\rho \sqrt{1 - \rho^2}}{\varphi_i} \frac{\rho^{2n+1}}{\sqrt{1 - \rho^2}} d\rho \underbrace{\int_{\mathbb{S}^{2n-1}} \frac{\partial(\mu^2)}{\partial \zeta} d\sigma_{\mathbb{S}^{2n-1}}}_{=0} = 0,$$

where we have used the Divergence Theorem for the Sphere \mathbb{S}^{2n-1} together with Lemma 2.16. This implies that

$$\int_{\mathbb{S}_{\mathbb{H}^n}} |\text{grad}_{HS} \mu|^2 \sigma_H^{2n} = (\lambda - \lambda_i) \int_{\mathbb{S}_{\mathbb{H}^n}} \mu^2 \sigma_H^{2n},$$

from which one obtains $\lambda \geq \lambda_i$. The last equation says that μ must be an eigenfunction of \mathcal{L}_{HS} on $\mathbb{S}_{\mathbb{H}^n}$ with eigenvalue $(\lambda - \lambda_i)$. By Lemma 2.20, the only possibility is $\lambda = \lambda_i$. It follows from Proposition 1.23 that μ must be constant.

¹³One uses also the following:

$$2 \langle \text{grad}_{HS} \varphi_i, \text{grad}_{HS} \mu \rangle = -2 \frac{\partial \varphi_i}{\partial \nu_H} \frac{\partial \mu}{\partial \nu_H} = -2 \left\langle (\varphi_i)'_\rho \frac{z}{\rho}, z + \kappa z^\perp \right\rangle \langle \nabla_{\mathbb{R}^n} \mu, z + \kappa z^\perp \rangle = -2\rho \sqrt{1 - \rho^2} (\varphi_i)'_\rho \mu'_\zeta.$$

The last remark shows how more general solutions cannot be found in a too simple way. A possible strategy could be to use multiple Fourier series.

Proposition 2.22. *The spectrum of Problem 2.4 is contained in the spectrum of Problem 1.29.*

Proof. Let φ be a smooth solution of Problem 1.29 and fix spherical coordinates $(\rho, \xi) \in [0, 1] \times \mathbb{S}^{2n-1}$. In particular, we are assuming that φ solves the equation $\mathcal{L}_{HS} \varphi = -\lambda \varphi$. Set $\varphi_0(\rho) := \int_{\mathbb{S}^{2n-1}} \varphi(\rho, \xi) d\sigma_{\mathbb{S}^{2n-1}}(\xi)$. Roughly speaking, φ_0 is the spherical mean of φ and hence a radial function.

At this point, the proof will immediately follow from the next:

Claim 2.23. *Let $\varphi_0 \neq 0$. Then the function φ_0 is an eigenfunction of Problem 2.4 associated with λ .*

Proof of Claim 2.23. First, note that

$$\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi_0 \sigma_H^{2n} = \int_{\mathbb{S}_{\mathbb{H}^n}} \int_{\mathbb{S}^{2n-1}} \varphi (d\sigma_{\mathbb{S}^{2n-1}} \wedge \sigma_H^{2n}) = \int_{\mathbb{S}^{2n-1}} \underbrace{\left(\int_{\mathbb{S}_{\mathbb{H}^n}} \varphi \sigma_H^{2n} \right)}_{=0} d\sigma_{\mathbb{S}^{2n-1}} = 0.$$

By making use of (20) we have

$$\begin{aligned} \mathcal{L}_{HS} \varphi &= (1 - \rho^2) \varphi''_{\rho\rho} + \frac{2n - (2n+1)\rho^2}{\rho} \varphi'_\rho - 2\rho\sqrt{1 - \rho^2} \varphi''_{\zeta\rho} \\ &\quad + \frac{1}{\rho^2} \Delta_{\mathbb{S}^{2n-1}} \varphi - (1 - \rho^2) \varphi''_{\zeta\zeta} - (Q-1)\sqrt{1 - \rho^2} \varphi'_\zeta. \end{aligned}$$

Integrating this expression along \mathbb{S}^{2n-1} , using Lemma 2.16 and the Divergence Theorem for the Sphere \mathbb{S}^{2n-1} , yields

$$\int_{\mathbb{S}^{2n-1}} \mathcal{L}_{HS} \varphi d\sigma_{\mathbb{S}^{2n-1}} = \int_{\mathbb{S}^{2n-1}} \left((1 - \rho^2) \varphi''_{\rho\rho} + \frac{2n - (2n+1)\rho^2}{\rho} \varphi'_\rho \right) d\sigma_{\mathbb{S}^{2n-1}}. \quad (27)$$

Furthermore, one has

$$\begin{aligned} \mathcal{L}_{HS} \varphi_0 &= (1 - \rho^2) \frac{\partial^2 \varphi_0}{\partial \rho^2} + \frac{2n - (2n+1)\rho^2}{\rho} \frac{\partial \varphi_0}{\partial \rho} \\ &= \int_{\mathbb{S}^{2n-1}} \left((1 - \rho^2) \varphi''_{\rho\rho} + \frac{2n - (2n+1)\rho^2}{\rho} \varphi'_\rho \right) d\sigma_{\mathbb{S}^{2n-1}}. \end{aligned}$$

So we finally get that

$$\mathcal{L}_{HS} \varphi_0 = \int_{\mathbb{S}^{2n-1}} \mathcal{L}_{HS} \varphi d\sigma_{\mathbb{S}^{2n-1}} = -\lambda \int_{\mathbb{S}^{2n-1}} \varphi d\sigma_{\mathbb{S}^{2n-1}} = -\lambda \varphi_0$$

which proves Claim 2.23. □

□

Remark 2.24 (Question). *If μ denotes the 1st eigenvalue of Problem 1.29, then is it true that $\mu = Q-1$? Roughly speaking, is the 1st eigenvalue of Problem 1.29 equal to the first eigenvalue of Problem 2.4?*

We now state a lemma which is well-known in the classical setting; see [14].

Lemma 2.25. *Let $S \subset \mathbb{H}^n$ be a hypersurface of class \mathbf{C}^2 and let $\Omega \subset S$ be any bounded open domain. If there exists a function $\psi > 0$ on Ω satisfying the equation $\mathcal{L}_{HS} \psi = q\psi$, then*

$$\int_{\Omega} (|\text{grad}_{HS} \varphi|^2 + q\varphi^2) \sigma_H^{2n} \geq 0$$

for all smooth function φ compactly supported on Ω .

For a proof, see [30].

Corollary 2.26. *Let $\Omega \in \mathbb{S}_{\mathbb{H}^n}^+$ or $\Omega \in \mathbb{S}_{\mathbb{H}^n}^-$. Then, the following inequality holds*

$$\int_{\Omega} (|\text{grad}_{\text{HS}} \varphi|^2 - (Q-1) \varphi^2) \sigma_H^{2n} \geq 0$$

for all smooth function φ compactly supported on Ω .

Proof. Setting $q = -(Q-1)$, the thesis follows by applying Lemma 2.25 with the choice $\psi = \sqrt{1-\rho^2}$ \square

Another easy consequence of Lemma 2.25 is contained in the next:

Corollary 2.27. *Let us set $\mathbb{S}^* := \left\{ p = \exp(z, t) \in \mathbb{S}_{\mathbb{H}^n} : \rho = |z| \geq \sqrt{\frac{Q-1}{Q}} \right\}$. Then, for every $\Omega \in \mathbb{S}^*$ the following inequality holds*

$$\int_{\Omega} (|\text{grad}_{\text{HS}} \varphi|^2 - 2Q \varphi^2) \sigma_H^{2n} \geq 0$$

for all smooth function φ compactly supported on Ω .

Proof. Let $q = -2Q$. Note that the function $\psi := \varphi_2(\rho) = Q\rho^2 - (Q-1)$ is strictly positive on every open subset $\Omega \in \mathbb{S}^*$. Since $\mathcal{L}_{\text{HS}} \psi = q\psi$, the thesis follows by applying Lemma 2.25. \square

In the inequalities of both Corollary 2.26 and Corollary 2.27, the function φ is not necessarily zero-mean. In other words, we do not require the validity of the condition $\int_{\Omega} \varphi \sigma_H^{2n} = 0$. Therefore, these inequalities are stronger than what one might expect: this seems to suggest a positive answer to the question stated in Remark 2.24. However, a multiple Fourier series approach seems unavoidable in order to give a rigorous proof.

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